\[ f(x) = \frac{2x^2}{x^3} \]

\[ g(x) = \cos x \]

\[ h(x) = -\cos x \]

\[ G(x) = C \quad \text{(cont. C)} \]

\[ F(x) = x^2 \]

\[ \text{Given } f(x), \text{ find } f'(x) \]

\[ \text{Find } y = f(x) \quad \text{Ch. 2+3} \]

\[ \text{Ch. 4} \quad \text{“Antiderivative”} \]

\[ \text{Ch. 4} \quad \text{“Antiderivative”} \]

\[ \text{Ch. 3} - \text{Derivative} \]

\[ \text{Chapters 1 - Limits} \]

\[ \text{MATH 1324} \]
\[ F(x) = x^2 + 1 \]
\[ F(x) = x^2 \quad \text{(a line)} \]
\[ F(x) = x^2 + e^x \quad \text{(always add a constant)} \]

**There are infinitely many antiderivatives.**

They all differ by a constant. The antiderivative is a family of functions.

\[ \text{A fn. } F(x) = \text{antiderivative of } f(x) \]

**Notation:**

- \( F(x) = \text{antiderivative of } f(x) \)
- \( F'(x) = \text{derivative of } F(x) \)
- \( F(x) = \text{original } f(x) \)

\[ F(x) = 2x \]

\[ F'(x) = 2 \cdot x \]
Theorem 4.1

If $F(x)$ is an antiderivative of $f(x)$ on an interval $I$, then $G(x) = F(x) + c$ for all $x \in I$.

All antiderivatives differ only by a constant.

If $G(x) = F(x) + c$, then $G(x)$ is an antiderivative of $f(x)$ on an interval $I$.

Proof:

1. If $B$ is true, then $A$ is true.
2. If $A$ is true, then $B$ is true.

Note:

In an "IFF" proof, we need to show $B$ is true if and only if $A$ is true.
Assume $G(x)$ is antiderivative of $f(x)$.

Show $G(x) = F(x) + C$.

(Proof by contradiction).

Let $H(x) = G(x) - F(x)$.

Assume $H(x) \neq C$ (not a constant).

Show this assumption leads to a contradiction.

$\Rightarrow H(x) = C$.

If $H$ is not constant, there must be 2 pts.

$A, b \in I$ s.t. $H(a) \neq H(b)$.

Mean Value Thm (MVT) says there is some pt.

$c \in (a, b)$ s.t.

$H'(c) = \frac{H(b) - H(a)}{b - a} \neq 0$.

But $H'(x) = \frac{d}{dx} [G(x) - F(x)] = f(x) - f(x) = 0$.

$H'(c) = f(c) - f(c) = 0$. 

Contradictory.
ex: Find the antiderivative of

$$f(x) = x^3$$

$$F(x) = \frac{1}{4} x^4 + C$$

Check:

$$\frac{d}{dx} \left[ \frac{1}{4} x^4 + C \right] = x^3$$

ex: Solve the differential egn

$$\frac{dy}{dx} = \frac{1}{4} x^4 + C$$

Soln.

$$y = \frac{1}{4} x^4 + C$$
Indefinite Integral Notation

Example: Evaluate $\int x^3 \, dx$

\[ \int x^3 \, dx = \frac{1}{4} x^4 + C \]

Note:
1. Find the antiderivative: $f(x) = \cos x$
2. Solve the differential equation: $\frac{dy}{dx} = \cos x$\[ \text{D.E.} \]
3. Evaluate: $\int \cos x \, dx$

Solution: $-\sin x + C$
Basic Integration Formulas.

First Five

1. \( \int 0 \, dx = C \)

2. \( \int k \, dx = kx + C \)
   (k is a constant)

3. \( \int k f(x) \, dx = k \int f(x) \, dx \)

4. \( \int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx \)
   (Integrate “term by term”)

5. *Power Rule* \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \)
   \( n \neq -1 \)
\[ \int (3x^2 - x) \, dx = \int 3x^2 \, dx - \int x \, dx = x^3 - \frac{1}{2}x^2 + C \]

\[ \int 2x \, dx = x^2 + C \]

\[ \int \frac{1}{2}x^2 \, dx = \frac{1}{6}x^3 + C \]

\[ \int \sqrt{x} \, dx = \frac{2}{3}x^{3/2} + C \]

\[ \int \sin x \, dx = -\cos x + C \]

\[ \int \cos x \, dx = \sin x + C \]

\[ \int \sec^2 x \, dx = \tan x + C \]

\[ \int \sec x \tan x \, dx = \sec x + C \]

\[ \int \csc^2 x \, dx = -\cot x + C \]

\[ \int \csc x \cot x \, dx = -\csc x + C \]
ex.

1. \[ \int (\theta^2 + \sec^2 \theta) \, d\theta = \frac{1}{3} \theta^3 + \tan \theta + C \]

2. \[ \int \sec \theta (\sec \theta - \tan \theta) \, d\theta = \int (\sec^2 \theta - \sec \theta \tan \theta) \, d\theta \]
   
   \[= \tan \theta - \sec \theta + C \]

3. \[ \int \frac{\sin x}{1 - \sin^2 x} \, dx = \int \frac{\sin x}{\cos^2 x} \, dx \]
   
   \[= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \, dx = \int \tan x \sec x \, dx \]
   
   \[= \sec x + C \]

Sometimes we can find the value of \( C \) in an antiderivative. In order to do this, you have to be given some additional info — it's called an initial condition.
Given $y(0) = 6$

$$y = \int \left( \frac{1}{3} x^3 + x \right) \, dx$$

$$= \frac{1}{3} x^4 + \frac{1}{2} x^2 + C$$

Plug in $x = 0$, $y = 6$

$$0 = \frac{1}{12} \cdot 0^4 + \frac{1}{2} (0)^2 + C$$

$$6 = \frac{1}{12} C$$

$$C = 72$$

$$y = \frac{1}{12} x^4 + \frac{1}{2} x^2 + 6$$

Solve.
Sec. 4.2  Finding Area under a curve.

(Leave briefly antiderivatives - find area - theorem that will tie the 2 together)

we can find the area of:

1. Rectangle: \( A = l \cdot w \)
2. Triangle: \( A = \frac{1}{2} \cdot b \cdot h \)
3. Circle: \( A = \pi r^2 \)

Goal use calculus to find area under a curve.

First we need notation.

**Summation Notation** \( \sum \)

\[ \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n \]
\[ i = \text{index of summation} \]

Subscript \((i, j, k)\) often used for indices

2 Properties of Finite Sums

\[ \sum_{l=1}^{n} a_{il} \text{ General Term} \]

Ex: Write in expanded form:

1. \[ \sum_{l=3}^{5} (2i_l) = (2 \cdot 3) + (2 \cdot 4) + (2 \cdot 5) \]

2. \[ \sum_{k=0}^{4} (k^2 - 1) = (0^2 - 1) + (1^2 - 1) + (2^2 - 1) + (3^2 - 1) + (4^2 - 1) \]

3. \[ \sum_{l=1}^{n} \frac{i}{n} = \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \ldots + \frac{n-1}{n} + \frac{n}{n} \]
1. $\sum_{i=1}^{n} k a_i = k \sum_{i=1}^{n} a_i$
2. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Find (Gauss)

$\sum_{i=1}^{100} (i) = \frac{n(n+1)}{2} = \frac{100(101)}{2} = 50 \cdot 101 = 5050$

$1 + 2 + 3 + \ldots + 99 + 100 = \sum_{i=1}^{100} (i) = \left( \frac{100}{2} \right) (101) = \frac{n}{2} (n+1)$
Formula

$$\sum_{l=1}^{n} i = \frac{n(n+1)}{2}$$

(Gauss)

1. Know this!

2. Know this

3. Know how to use 3 + 4 (do not memorize)

4. Know this:

   $$\sum_{l=1}^{n} l^2 = \frac{n(n+1)(2n+1)}{6}$$

5. $$\sum_{l=1}^{n} l^3 = \frac{n^2 (n+1)^2}{4}$$
Evaluate the sum:

1. $\sum_{i=1}^{10} \frac{1}{i^2} = 5 \cdot \frac{8}{9} \cdot 8 + 1.85 = 5 \\
2. \sum_{j=1}^{12} \left[ \frac{5}{j^2} \right] = \frac{1}{12} \left( \frac{12}{2} \cdot 7 \cdot \left( \frac{12+1}{2} \right) \right) + \frac{4 \cdot 12}{2} = \frac{1}{12} \left[ 7 + 12(12+1) \right] + 4 \cdot 12 = \frac{1}{12} \cdot 7 + 12(12+1) + 4 \cdot 12 = 14 + 156 = 170$