3.1 Extrema on an Interval

- Def Extrema

Let \( f \) be defined on an interval \( I \) containing \( c \).

\[
\begin{align*}
\mathbb{R} & \rightarrow [a,b] \quad \mathbb{R} & \rightarrow (a,b) \\
\mathbb{R} & \rightarrow (a,b) \quad \mathbb{R} & \rightarrow [a,b]
\end{align*}
\]

1. \( f(c) \) is the minimum of \( f \) on \( I \) when \( f(x) \geq f(c) \) for all \( x \in I \).
2. \( f(c) \) is the maximum of \( f \) on \( I \) when \( f(x) \leq f(c) \) for all \( x \in I \).

- min/max values of a function over \( I \) are called \text{extremum}.

Singular of extremum is extremum.

- min/max of a function over \( I \) implies \text{Absolute min/max}.

Not always going to have a min/max on \( \text{EVERY} \) Interval!

Thm 3.1 The Extreme Value Theorem

If \( f \) is continuous on a closed interval \([a,b]\), then \( f \) has both a minimum and a maximum on the interval.

\( \ast \) They \text{ EXIST!} No clue what they are... but they're something!

- Def Relative Extrema

1. If there is an open interval containing \( c \) on which \( f(c) \) is a max,
then \( f(c) \) is called a relative max of \( f \), can say "\( f \) has relative max at \( c \)."

2. If there is an open interval containing \( c \) on which \( f(c) \) is a min,
then \( f(c) \) is called a relative min of \( f \), can say "\( f \) has relative min at \( c \)."

- Plural is maxima & minima for relative/local.

- Def Critical Number

Let \( f \) be defined at \( c \). If \( f(c) = 0 \) \text{ OR } if \( f \) is not differentiable at \( c \), then \( c \) is a \text{critical number} of \( f \).

\[ \text{Fig 3.4} \]
Theorem 3.2: Relative Extrema Occur Only at Critical Numbers

If \( f \) has a relative min/max at \( x = c \), then \( c \) is a critical number of \( f \).

Guidelines for Finding Extrema on a Closed Interval

1. Find the critical numbers of \( f \) in \( (a, b) \).
2. Evaluate \( f \) at each critical number in \( (a, b) \). “Plug in critical numbers.”
3. Evaluate \( f \) at each endpoint of \( [a, b] \). “Plug in a & b.”
4. The smallest value of step 2 & 3 is the minimum (absolute/global minimum).
   The largest value of step 2 & 3 is the maximum (absolute/global maximum).

**Example 1:**

\[ f(x) = 3x^4 - 4x^3 \] on \([1, 2] \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
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<tbody>
<tr>
<td>Left End Pt.</td>
<td>-1</td>
</tr>
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<td></td>
<td>0</td>
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<td></td>
<td>1</td>
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<td>2</td>
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Absolute Min: \((0, 0)\)

Absolute Max: \((2, 16)\)

**Example 2:**

\[ f(x) = 2 - 3x^{2/3} \] on \([-1, 3]\)

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<td>3</td>
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Absolute Min: \((-1, -5)\)

Absolute Max: \((0, 0)\)

**Example 3:**

\[ f(x) = 2 \sin(x) - \cos(2x) \] on \([0, 2\pi]\)

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