Sec. 3.3  The First Derivative Test

- Review
- Read graphs

\[ \begin{align*}
\text{Inc.:} & \quad \text{Increasing} \\
\text{Dec.:} & \quad \text{Decreasing} \\
\end{align*} \]

"Turning points" \Rightarrow \text{Critical numbers}

**Define**

1. A fn. \( f(x) \) is \text{Inc.} on an interval \( I \) when for any \( x_1, x_2 \) in \( I \)
   \[ f(x_1) < f(x_2) \]
2. A fn. \( f(x) \) is dec. on an interval \( I \) when for any \( x_1 < x_2 \) in \( I \)
\[
f(x_1) > f(x_2)
\]

**Thm 3.5** Let \( f(x) \) be a cont. fn. on \([a,b]\) and diff. on \((a,b)\). Then

1. If \( f'(x) > 0 \) for all \( x \in (a,b) \) then \( f(x) \) is inc. on \((a,b)\).
2. If \( f'(x) < 0 \) for all \( x \in (a,b) \) then \( f(x) \) is dec. on \((a,b)\).
3. If \( f'(x) = 0 \) for all \( x \in (a,b) \) then \( f(x) \) is horizontal constant on \((a,b)\).

To find where a fn. \( f(x) \) is inc./dec.

1. Find the critical numbers of \( f \) on \((a,b)\).
   - Find \( f'(x) \) and \( f'(x) = 0 \) or one.
Plot critical numbers on a number line.

1. Pick a test pt. from each interval on the number line.
2. Plug (sub.) test pt. into f'(x).
   - If f'(x) > 0, f(x) is inc.
   - If f'(x) < 0, f(x) is dec.
3. Write ans. as open intervals.

Example: Determine the open intervals on which f(x) is inc. or dec.

f(x) = x^4 - 32x + 4
f'(x) = 4x^3 - 32 = 0  (always defined)

4(x^3 - 8) = 0
x^3 - 8 = 0
x^3 = 8
x = \sqrt[3]{8} = 2
Problem

The open intervals on which

\[ f(x) = \begin{cases} x^3 - 6x^2 = 0 & \text{inc.} \\
2x^2 - 3 = 0 & \text{dec.} \\
x = 0 \end{cases} \]

Find

- \( f(x) = \begin{cases} 3x - 2 & \text{inc.} \\
2x^2 = 0 & \text{dec.} \\
x = 3/2 \end{cases} \)

Ans.

\[ x = 0 \]

\[ x = 3 \]

\[ x = 3/2 \]
The First Derivative Test

Let \( c \) be a critical number of a continuous function \( f(x) \) on an open interval \( I \) containing \( c \). If \( f(x) \) is differentiable on \( I \) except possibly at \( x = c \), then \( f(c) \) is
(1) If \( f'(x) \) changes from neg. to pos. at \( x = c \), then \( x = c \) is a relative max.

(2) If \( f'(x) \) changes from pos. to neg. at \( x = c \), then \( x = c \) is a relative min.

(3) If \( f'(x) \) does not change direction at \( x = c \), then \( x = c \) is neither a max. nor a min.
Find relative extrema. To find relative extrema, use the First Derivative Test.

Ex.

\( f(x) = x^3 - 6x^2 + 9x \)

\( f'(x) = 3x^2 - 12x + 9 \)

To find relative extrema, set the derivative equal to zero:

\( 3x^2 - 12x + 9 = 0 \)

Solve for \( x \):

\( x = 0, 1, \frac{3}{2} \)

Use the First Derivative Test.
\[ f(x) = x^4 - 2x^3 f(1.5) = (1.5)^4 - 2(1.5)^3 = -1.69 \]

Find open intervals of \( \text{Inc.} \text{ Decl.} \),

\[ f'(x) = 4x^3 - 6x^2 \]

\[ f'(x) = 0 \]

\[ x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi \]

\[ \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \pi \]

\[ x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \pi \]

\[ f(x) = 1 + 2\cos x = 0 \]

\[ \cos x = -\frac{1}{2} \]

\[ \theta = \frac{\pi}{3} \]
\[ f'(\pi/2) = 1 + 2 \cos(\pi/2) = 1 \cdot \text{pos}. \]
\[ f'(\pi) = 1 + 2 \cos(\pi) = -1 \cdot \text{neg}. \]
\[ f'(3\pi/2) = 1 + 2 \cos(3\pi/2) = 1 \cdot \text{pos}. \]

\[
\begin{align*}
\text{inc.} & : (0, \frac{2\pi}{3}) \quad \left(\frac{4\pi}{3}, 2\pi\right) \\
\text{dec.} & : \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)
\end{align*}
\]

\[ \text{rel. extrema} \]
\[ \text{rel. max at } x = \frac{2\pi}{3} \quad y = f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sqrt{3} \]
\[ \text{rel. min. at } x = \frac{4\pi}{3} \quad y = f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3} \]
Sec. 3.4 Concavity and the Second Derivative Test

Goal: What does \( f''(x) \) [2nd Der.] tell us about the fn. \( f(x) \)?

Concavity

- Graph is concave up — "holds water" "curving up"
- Graph is concave down — "spills water" "curving down"
Define let \( f(x) \) be diff. on an open interval \( I \).

1. The graph of \( f(x) \) is **concave up** on \( I \) when \( f'(x) \) is inc. on \( I \).
   \[ f''(x) > 0 \] 

2. The graph of \( f(x) \) is **concave down** on \( I \) when \( f'(x) \) is dec. on \( I \).
   \[ f''(x) < 0 \]

**Theorem 3.7 Test for Concavity**

Let \( f(x) \) be a fn. whose 2nd derivative exists on the open interval \( I \).

1. If \( f''(x) > 0 \) on \( I \), the graph of \( f(x) \) is concave up on \( I \).
2. If \( f''(x) < 0 \) on \( I \), the graph of \( f(x) \) is concave down on \( I \).
Steps

1. Find $f''(x) = 0$ or DNE
2. Find where $f''(x)$ changes from $<$ to $>$ or from $>$ to $<$.
3. Plot a test pt. from each interval on the line. The function will be c. up or c. down.
4. $f''(x) > 0$ c. up
5. $f''(x) < 0$ c. down

Ex: $y = f(x) = -x^3 + 3x^2 - 2$.
$f(x) = -3x^2 + 6x$.
$f'(x) = 6x + 6 = 0$.
$x = -1$.
$f''(x) = -3x^2 + 6x$.
\[ c. \text{up} \quad | \quad c. \text{down} \]

\[ x = 1 \]

\[ x = 0 \quad f''(0) = -6(0) + 6 = 6 > 0 \quad c. \text{up} \]

\[ x = 2 \quad f''(2) = -6(2) + 6 = -6 < 0 \quad c. \text{down} \]

\[ c. \text{up} \]

\[ (-\infty, 1) \quad c. \text{down} \quad (1, \infty) \]

\[ x = 1 \text{ is an inflection point} \] - this is a pt. where the fn. changes concavity.