1. To find the nth term of a geometric series
2. To sum a geometric series

\[ 1, 2, 4, 8, 16, 32, 64 \]

**Recursive**: Multiply by 2 to get the next term

\[
\frac{64}{32} = 2, \quad \frac{8}{4} = 2
\]

In a **geometric sequence**, a term is found by multiplying the previous term by a common ratio, \( r \). (so \( r=2 \) in the above example)

So, recursively, \( a_n = a_{n-1} \cdot r \)

An *Explicit Formula* for \( a_n \) (geometric sequence)

\[
a_1 = a_1 \quad a_2 = a_1 \cdot r
\]
\[ a_2 = (a_1 \cdot r) \]
\[ a_3 = a_2 \cdot r = (a_1 \cdot r) \cdot r = a_1 \cdot r^2 \]
\[ a_4 = a_3 \cdot r = (a_1 \cdot r^2) \cdot r = a_1 \cdot r^3 \]
\[ a_5 = a_4 \cdot r = a_1 \cdot r^4 \]
\[ \vdots \]
\[ a_n = a_1 \cdot r^{n-1} \]

Explicit Formula for a Geometric Sequence

\((\text{Ex})\) Find \(a_5\), \(a_9\), and \(a_n\).

\[ a_1 = -3, 9, -27, \ldots \]
\[ r = \frac{-3}{9} = -\frac{1}{3} \]
\[ a_5 = (-27)(-3) = 81 \]

Recursive

Explicit

\[ a_n = a_1 \cdot r^{n-1} \]
\[ a_n = 1(-3)^{n-1} \]
\[ a_n = (-3)^{n-1} \]

\[ a_9 = (-3)^9 = (-3)^6 = 6561 \]

Find a formula for the sum of a geometric sequence.

\[ S_n = \sum_{k=1}^{n} a_k \]

\[ S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} \]

\[ -rS_n = -a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^n \]

\[ S_n - rS_n = a_1 - a_1r^n \]

\[ S_n \frac{(1-r)}{1-r} = a_1 \frac{(1-r^n)}{1-r} \]

\[ S_n = a_1 \frac{(1-r^n)}{1-r} \]

Find the sum.

a) \[ \sum_{n=0}^{6} \frac{2}{3}^n = S_6 \]
\[ a_1 = 2 \cdot (\frac{2}{3})^0 = 2 \]

\[ r = 5 \]

\[ n = 8 \]

\[ S_n = \frac{a_1(1 - r^n)}{1 - r} \]

\[ S_6 = \frac{2 \cdot (1 - (\frac{2}{3})^6)}{1 - \frac{2}{3}} \]

\[ = \frac{2 \cdot (1 - (\frac{2}{3})^6)}{\frac{1}{3}} \]

\[ = 2 \cdot (1 - (\frac{2}{3})^6) \]

\[ = \frac{1330}{729} \]

\[ b) \sum_{n=0}^{7} 2 \cdot (5)^n \]
\[ r = 5 \]
\[ n = 8 \]
\[ S_8 = \frac{2(1 - 5^8)}{1 - 5} = \frac{2(1 - 5^8)}{-4} \]
\[ = -(1 - 5^8) \]
\[ = 195312 \]

**Question:**

How many “parents” (parents, grandparents, great-grandparents, etc.) do you have going back 10 generations?

\[ a_1 = 2 \]
\[ r = 2 \]
\[ n = 10 \]
\[ S_{10} = \frac{2(1 - 2^{10})}{1 - 2} \]
\[ = -2(1 - 2^{10}) \]
\[ = 2046 \]
\[ S_{20} = -2(1 - 2^{20}) \]
\[ = 2046 \]
\[ n = 20 \text{ generations} \]

**Compound Interest**

- \( P \) = principal (initial amount of \$ invested)
- \( r \) = yearly interest rate as a decimal
- \( n \) = number of interest compounding periods per year
- \( t \) = time in years
- \( A \) = amount of money earned after time \( t \)

\[
A = P(1 + \frac{r}{n})^{nt}
\]

Assume \( n = 1 \)

- \( A_0 = P \)
- \( A_1 = P(1+r) \)
- \( A_2 = P(1+r)(1+r) = P(1+r)^2 \)
Ex. Find the amount of money in an account after 5 years if $12,500 is invested initially with an annual interest rate of 2.5% if the compounding is done (a) semi-annually and (b) quarterly.

\[ A_5 = \rho (1 + \frac{r}{n})^{nt} \]

(a) \[ n = 2, \quad A_5 = 12,500 \left(1 + \frac{0.025}{2}\right)^{2 \cdot 5} = $14,153.39 \]

(b) \[ n = 4, \quad A_5 = 12,500 \left(1 + \frac{0.025}{4}\right)^{4 \cdot 5} = $14,158.84 \]

How to become a millionaire (theoretically)
Invest $5500 per year via a Roth IRA into the stock market.

\[ A_n = \rho (1 + r)^t \]

<table>
<thead>
<tr>
<th>Deposit #</th>
<th>Value of the deposit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$5500 (1 + 0.08)^{18}</td>
</tr>
<tr>
<td>2</td>
<td>$5500 (1.08)^{17}</td>
</tr>
<tr>
<td>n=1</td>
<td>geometric sequence</td>
</tr>
</tbody>
</table>
Section 11.3 Geometric Sequences and Series Page 8

2
3
18

geometric sequence
\[ r = 1.08 \]

Amount of money after 18 yrs is the sum of 2nd column

\[
\begin{align*}
S_{18} &= 5500 (1.08)^{18} \\
&= 222,454
\end{align*}
\]

Use \( S_n = \frac{a_1 (1-r^n)}{1-r} \) to estimate amount of money after 18 years.

Use \( S_n = \frac{a_1 (1-r^n)}{1-r} \) to estimate amount of money after 18 years.

Sum of an infinite geometric series

\[
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2
\]

\[
S_n = \frac{a_1 (1-r^n)}{1-r}, \quad \text{note } r \to 0 \text{ as } n \to \infty \]

\[
\lim_{n \to \infty} a_1 (1-r^n) = a_1, \quad -1 < r < 1
\]
\[ \lim_{{n \to \infty}} \frac{{a_1 (1 - r^n)}}{{1 - r}} \rightarrow \frac{{a_1}}{{1 - r}}, \quad -1 < r < 1 \]

\[ S = \frac{{a_1}}{{1 - r}}, \quad -1 < r < 1 \]

\textbf{b) Find the sum:} \[ 1 + \frac{1}{2} + \frac{1}{4} + \cdots \]

\[ r = \frac{1}{2}, \quad a_1 = 1 \]

\[ S = \frac{{a_1}}{{1 - r}} = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2 \]

\textbf{b) } \sum_{{n=0}}^{{\infty}} \left( \frac{7}{10} \right)^n

\[ a_1 = \left( \frac{7}{10} \right)^0 = 1, \quad -1 < r = \frac{7}{10} < 1 \]

\[ S = \frac{{a_1}}{{1 - r}} = \frac{1}{1 - \frac{7}{10}} = \frac{1}{\frac{3}{10}} = \frac{10}{3} \]
\( \text{claim: } 0.\overline{9} = 1 \)

\[ 0.\overline{9} = 0.99999\ldots \]

\[ = 0.9 + 0.09 + 0.009 + 0.0009 + \ldots \]

\[-1 < r = 0.1 < 1, \quad a_1 = 0.9 \]

\[ S = \frac{a_1}{1-r} = \frac{0.9}{1-0.1} \]

\[ = \frac{0.9}{0.9} \quad \square \]

\( \text{write as a rational number:} \)

\[ 0.\overline{63} \]

\[ .\overline{63} = 0.636363636363 \]

\[ = 0.63 + 0.0063 + 0.000063 + \ldots \]

\[ r = 0.01, \quad a_1 = 0.63 \]

\[ S = \frac{a_1}{1-r} = \frac{0.63}{1-0.01} \]
\[ 1 - 0.01 = \frac{0.63}{0.99} = \frac{63}{99} = \left(\frac{7}{11}\right) \]

\[ \text{ex} \quad 0.0\overline{123} \]