**Section 3.1: The Remainder and Factor Theorems**

**Definition:** A polynomial function has the form

\[ p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]

- **Example 1:** \( p(x) = c \), \( c \) a constant

- **Example 2:** \( p(x) = m x + b \) (linear polynomial function), \( \text{deg} = 1 \)

- **Example 3:** \( p(x) = ax^2 + bx + c \) (quadratic polynomial, \( \text{deg} = 2 \))

**Vertex Formulas**

- **Long Divide**
  
  a) \[ \frac{x^2 + 4x - 8}{x - 3} = x + 7 + \frac{13}{x - 3} \]

  
  x - 3 \[ \overline{x^2 + 4x - 8} \]
  
  \[ \begin{array}{r}
  x + 7 \\
  \hline
  x^2 + 4x - 8 \\
  - (x - 3)(x + 7) \\
  \hline
  13
  \end{array} \]

**Synthetic Division**

Works when the divisor is of the form \( x - c \), \( c \) a constant.
The Remainder Theorem

If R is the remainder when P(x) is divided by x-c, then R=P(c).

\( P(x) = x^3 - 3x^2 + x + 1 \)

a) Find the remainder when \( P(x) \) is divided by \( x+2 \)

\[
\begin{array}{c|cccc}
-2 & 1 & -3 & 1 & 1 \\
   &   & -2 & 10 & 11 \\
\hline
   & 1 & -5 & 11 & R = 21
\end{array}
\]

Q: \( x^2 - 5x + 1 \)
R: \(-21\)

b) Find \( P(-2) \)

\[
P(-2) = (-2)^3 - 3(-2)^2 + (-2) + 1
\]

\[
= -8 - 12 - 2 + 1
\]

\[
= -21
\]

Def: A number \( c \) is a zero of \( P(x) \) iff \( P(c) = 0 \)
The Factor Theorem

The number c is a zero of a polynomial, \( P(x) \), iff \( x - c \) is a factor of \( P(x) \).

Note: Real zeros of \( P(x) \) correspond to \( x \)-intercepts.

\[ \text{c is a zero and the x-int. is } (c, 0). \]

Ex. Find the zeros of

\[ P(x) = x^2 - x - 6 \]

\[ 0 = x^2 - x - 6 \]

\[ 0 = (x + 2)(x - 3) \]

\[ x + 2 = 0 \quad \text{or} \quad x - 3 = 0 \]

\[ x = -2 \quad \text{or} \quad x = 3 \]

Zeros of \( P(x) \)

Ex. Let \( P(x) = 3x^4 + 8x^3 + 10x^2 + 2x - 20 \)

a) show \( c = -2 \) is a zero of \( P(x) \)

\[
\begin{array}{c|cccc}
-2 & 3 & 8 & 10 & 2 -20 \\
 & -6 & 4 & -12 & \hline 20 \\
\hline
3 & 2 & 6 & -10 & 0
\end{array}
\]

So, \( -2 \) is a zero of \( P(x) \) by Remainder Theorem.

Q: \( 3x^3 + 2x^2 + 6x - 10 \)

Divisor: \( x - (-2) = x + 2 \)

b) factor \( P(x) \) into a linear factor times a cubic factor

\[ 3x^4 + 8x^3 + 10x^2 + 2x - 20 = (x + 2)(3x^3 + 2x^2 + 6x - 10) \]
Def: Suppose \( f(x) \) is a function.

The difference quotient of \( f \) is
\[
\frac{f(x+h) - f(x)}{h}, \quad h \neq 0
\]

\( m \) between \((x, f(x))\) and \((x+h, f(x+h))\) is the O.Q.

\[
O.Q. = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{x+h - x} = \frac{f(x+h) - f(x)}{h}
\]

ex Find the difference quotient:
\[
f(x) = x^2 + 5
\]

\[
f(x+h) = (x+h)^2 + 5
\]

\[
O.Q. = \frac{f(x+h) - f(x)}{h}
\]

\[
= \frac{(x+h)^2 + 5 - (x^2 + 5)}{h}
\]

\[
= \frac{x^2 + 2xh + h^2 + 5 - x^2 - 5}{h}
\]

\[
= \frac{2xh + h^2}{h}
\]
\[ h = \frac{K(2x+h)}{h} \]

= \[ 2x + h \]