Goal: To find the zeros of a polynomial

Theorem (FTA): Any polynomial of degree n with complex coefficients has exactly n zeros, counting multiplicities.

\[ P(x) = x^4 - 3x^3 - 4 \]

Factors of -4: \( \pm \{1, 2, 4\} \)

Factors of 1: \( \pm 1 \)

\( \text{RZs:} \ \pm \{1, 2, 4\} \) (suspects)

(Expect \( n = 4 \) zeros by FTA)

\[ \begin{array}{c|ccccc}
2 & 1 & 0 & -3 & 0 & -4 \\
-2 & 1 & 2 & 1 & 2 & 4 \\
\hline
1 & 0 & 1 & \text{LO} \\
\end{array} \]

\[ x^2 + 1 = 0 \]

Quotient contains remaining zeros

\[ x = \pm i \]
\[ x = \pm 2 \pm \pm i \]

**Zeros:** \((x = \pm 2, \pm i)\)

1. **fct:** \(p(x) = (x-2)(x+2)(x-i)(x+i)\)

2. **fct:** \(p(x) = 4x^3 - 10x^2 + 4x + 5\)

   **factors of 5:** \(\pm \{1, 5\}\)

   **factors of 4:** \(\pm \{1, 2, 4\}\)

   **factors:** \(\pm \{\pm 1\}, \frac{1}{4}, \frac{5}{4}\)

   **Roots:** \(\pm \{\pm 1\}, \frac{1}{4}, \frac{5}{4}\)

   \(\sqrt{2}\)

   **There are 3 zeros, by FTA.**

\[
\begin{array}{cccc}
4 & -10 & 4 & 5 \\
-2 & 6 & -5 \\
\hline
4 & -12 & 10 & 0
\end{array}
\]

\[4x^2 - 12x + 10 = 0\]

\[2x^2 - 6x + 5 = 0\]

\[x = \frac{6 \pm \sqrt{36 - 4(2)(5)}}{4}\]

\[x = \frac{6 \pm \sqrt{-4}}{4}\]

\[x = \frac{6 \pm 2i}{4}\]
\[
x = \frac{\frac{\frac{4}{2}}{\frac{2}{4}}}{\frac{3\pm i}{\frac{2}{i}}}, \quad -\frac{1}{2}
\]

Zeros: \(\frac{3}{2}, -\frac{1}{2}, \frac{3}{2} + \frac{1}{2}i, -\frac{1}{2}i\)  

Com|plex  
Conjugates

\[
P(x) = 4(x + \frac{1}{2})(x - (\frac{3}{2} - \frac{1}{2}i))(x - (\frac{3}{2} + \frac{1}{2}i))
\]

Conjugate Pairs Theorem

If \(a + bi\) is a zero of \(P(x)\), and \(P(x)\) has real coefficients, then \(a - bi\) is also zero.

**STUDY THIS**

**(Ex)** Let \(P(x) = x^4 - 2x^3 - x^2 + 6x - 6\).

Given \(1 - i\) is a zero of \(P(x)\), find the rest.

\[
\begin{array}{cccccc}
1 - i & 1 & -2 + 0i & -1 & 6 & -6 \\
 & 1 - i & -2 & -3 + 3i & 6 \\
1 + i & 1 & -1 - i & -3 & 3 + 3i & 10 \\
 & 1 + i & 0 & -3 - 3i & \\
\end{array}
\]
Given $1+2i$ and $3$ are zeros of a cubic polynomial with real coefficients, find an equation for that polynomial.

Since the coefficients are real, by CPT, $1-2i$ is also a zero.

Zeros: $1+2i, 1-2i, 3$

$p(x) = (x-3)(x-(1+2i))(x-(1-2i))$

$= (x-3)[x^2 - x(1-2i) + (1+2i)(1-2i)]$

$= (x-3)(x^2 - x + 2x + x - 2 + 1 - 4x^2)$

$= (x-3)(x^2 - 2x + 5)$

$= x^3 - 2x^2 + 5x - 15$
How would you find the zeros of 

\[ x^5 + 32 = 0 \] ?

\[ \sqrt[5]{x} = \sqrt[5]{-32} \]

\[ x = -2 \]

\[ \text{Time-out (Back to trig land)} \]

\[ z = a + bi = r(\cos \theta + i \sin \theta) \]

\[ = r \text{cis} \theta \]

\[ \text{trig form of } z, \text{ where } r = 1 \text{ZI} \]

\[ \text{Demoivre} \]

\[ z^n = r^n \text{cis}(n\theta) \]

\[ \text{Recall: } 2^{\frac{1}{4}} = \sqrt[4]{2}, a^{\frac{1}{n}} = \sqrt[n]{a} \]

\[ z^{\frac{1}{n}} = r^{\frac{1}{n}} \text{cis}\left(\frac{\theta}{n}\right) \]

\[ \text{Find all zeros of } x^5 + 32 = 0 \]
(i.e. find the five fifth roots of -32)

Write -32 in trig form.

\[ z = -32 + 0i \]
\[ z = r \text{cis} \theta \]
\[ r = 32 \]
\[ z = 32 \text{cis} 180^\circ \]

\[ z^n = r^n \text{cis}(n\theta) \]
\[ z^{\frac{1}{n}} = r^{\frac{1}{n}} \text{cis} \left( \frac{\theta}{n} \right) \]

\[ \theta + \frac{360^\circ k}{n} \]
\[ W_k = r^{\frac{1}{n}} \text{cis} \left( \frac{\theta + 360^\circ k}{n} \right) \]
\[ k = 0, 1, 2, \ldots, n-1 \]

\[ z^{\frac{1}{5}} = 32^{\frac{1}{5}} \text{cis} \left( \frac{180^\circ}{5} \right) = 2 \text{cis} 36^\circ \]

In the lingo from above

what I just found is …..

\[ W_0 = 2 \text{cis} 36^\circ \]
\[ W_1 = 2 \text{cis} \left( 36^\circ + 72^\circ \right) = 2 \text{cis} 108^\circ \]
\[ W_2 = 2 \text{cis} 180^\circ \]

\[ \frac{360^\circ}{n} = \frac{360^\circ}{5} = 72^\circ \]

\[ n = 5 \] not other
\[ w_2 = 2 \, \text{cis} \, 180^\circ \]
\[ w_3 = 2 \, \text{cis} \, 252^\circ \]
\[ w_4 = 2 \, \text{cis} \, 324^\circ \]

These are the zeros of \( x^5 + 32 = 0 \)

\{ \text{keep adding } 72^\circ \text{ to get other zeros (roots)} \}