Section 7.3: Vectors

Goals:
1. To +, -, • vectors
2. To write vectors in component form and \( \hat{i}, \hat{j} \) form
3. To solve apps

All of these "vectors" are equivalent in the sense that they point in the same direction and are the same length. We symbolize a vector in what looks like an ordered pair:
\[
\langle x, y \rangle
\]
In the above example, the vector is represented by \( \langle 1, 2 \rangle \). called component form of a vector. In the example, the horizontal component is \( x = 1 \) and the vertical component is \( y = 2 \).

Def: A vector is a directed line segment.

Scalar multiplication

\[
\text{red vector} = 1.5 \vec{v}
\]
Scalar

\[ -1.5 \vec{v} \] has the same length as \( 1.5 \vec{v} \) but opposite direction (due to the negative)

We say we "scaled" \( \vec{v} \) and made it 1.5 times as long. So, this kind of multiplication is called scalar mult.
and 1.5 is a scalar.
Symbolic Scalar Multiplication

\[ \vec{v} = c \vec{a} \]

black vector: \(1.5 \vec{a}\)

Note: if \(\vec{a} \approx \vec{b} \approx \vec{c}\)

1. So, there is a constant \(c\) of proportionality. In this case \(c = 1.5\).
2. Component form of \(1.5 \vec{a} = \langle 1.5a, 1.5b \rangle\)
   (see pic to left)

So, in general, \(c \vec{a} = \langle ca, cb \rangle\).

Symbolic scalar multiplication.

in the same direction as \(\vec{v}\). If \(k < 0\), \(k \vec{v}\) points in opposite direction to \(\vec{v}\).

Addition of Vectors (triangle law)

\[ \vec{v} = \langle 4, 5 \rangle \]
\[ \vec{u} = \langle 1, 4 \rangle \]
\[ \vec{v} + \vec{u} = \langle 5, 9 \rangle \]
**Def** Symbolic way to add vectors
\[ \vec{v} = \langle v_1, v_2 \rangle, \quad \vec{w} = \langle w_1, w_2 \rangle \]
\[ \vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle \]

**Ex** Let \( \vec{v} = \langle 2, 3 \rangle \) and \( \vec{w} = \langle 3, -4 \rangle \).
Perform the following operations

a) \( \vec{v} + \vec{w} \)
\[ = \langle 1, -1 \rangle \]

b) \( \vec{v} - \vec{w} \)
\[ = \vec{v} + (-\vec{w}) \]
\[ = \langle 2, 3 \rangle + \langle 3, -4 \rangle \]
\[ = \langle 5, 1 \rangle \]

c) \( 2\vec{w} \)
\[ = \langle 6, -8 \rangle \]

\[ \{ \text{This is called} \]
\[ \text{scalar multiplication} \]
\[ \text{where} \ 2 \ \text{is the} \]
\[ \text{scalar} \]

\[ d) \ 2\vec{w} + 3\vec{v} \]
\[ = \langle 6, -8 \rangle + \langle 6, 9 \rangle \]
\[ = \langle 12, 1 \rangle \]

**Magnitude and Direction**

Assumption:
- Tail at \((0,0)\)
- \(\theta\) is the}
\[ \theta = \arctan \left( \frac{b}{a} \right) \]

and \( \theta \) is the angle formed between positive \( x \)-axis and vector.

\( a \times \theta \) Let \( \mathbf{u} = \langle -3, 5 \rangle \). Find \( \| \mathbf{u} \| \) and \( \theta \)

\[ \| \mathbf{u} \| = \sqrt{(-3)^2 + 5^2} = \sqrt{34} \]

\[ \tan \theta = -\frac{5}{3} \]

\[ \theta = 121.0^\circ \]

\( \therefore \) Vectors in component form

a) \( \mathbf{v} = \langle 2, 3 \rangle \)

by the book: tail=(0,0)
\( \vec{v} \) by the book: tail: \((0,0)\)

b) \( \vec{u} = \langle -3, -4 \rangle \)

by book

Here, the initial point of \( \vec{v} \) is \((0,0)\) and the end point is \((2,3)\)

\( \Box \) Write the vector with tail at \( \text{tail} P(1,3) \) and tip at \( \text{tip} Q(3,5) \) in component form

Vector \( \overrightarrow{PQ} \)

\[ \text{tip} - \text{tail} \]
\[ \langle 3-1, 5-3 \rangle \]

\[ = \langle 2, 2 \rangle \]

\( \hat{i}, \hat{j} \) form

\( \Box \) Consider \( \langle 5,8 \rangle = \langle 5,0 \rangle + \langle 0,8 \rangle \)

\[ = 5 \langle 1,0 \rangle + 8 \langle 0,1 \rangle \]

\( \hat{i} \)
\( \hat{j} \)

\( \star \) Any vector \( \langle a, b \rangle \) can be written as \( a\hat{i} + b\hat{j} \) (\( \langle a, b \rangle = a\hat{i} + b\hat{j} \))
\[ \text{ex. } \text{ Let } \vec{u} = 3\hat{i} - 2\hat{j}, \quad \vec{v} = -5\hat{i} - 7\hat{j} \]

Find

a) \(2\vec{u} - 4\vec{v}\)
\[
= 2(3\hat{i} - 2\hat{j}) - 4(-5\hat{i} - 7\hat{j})
\]
\[
= 6\hat{i} - 4\hat{j} + 20\hat{i} + 28\hat{j}
\]
\[
= 26\hat{i} + 24\hat{j}
\]

b) \(\|\vec{v}\|\)
\[
\|\vec{v}\| = \sqrt{(-5)^2 + (-7)^2}
\]
\[
= \sqrt{74}
\]

Notes: Any vector \(\vec{v}\) with magnitude \(\|\vec{v}\|\) and angle \(\theta\) can be written in component form...

\[\langle \|\vec{v}\| \cos \theta, \|\vec{v}\| \sin \theta \rangle = \|\vec{v}\| \cos \theta \hat{i} + \|\vec{v}\| \sin \theta \hat{j}\]
\[ \mathbf{v} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} \]

**Problem 2:** \( \| c \mathbf{v} \| = |c| \| \mathbf{v} \| \), \( c \) a constant

**Problem 3:** \( \hat{\mathbf{v}} = \frac{\mathbf{v}}{\| \mathbf{v} \|} \) is a unit vector

**Proof:**

\[
\hat{\mathbf{v}} = \frac{\mathbf{v}}{\| \mathbf{v} \|}
\]

**Ex:** Find a vector \( \) unit long that points in the same direction as \( \mathbf{v} = \langle -3, -4 \rangle \)

\[
\| \mathbf{v} \| = \sqrt{(-3)^2 + (-4)^2} = 5
\]

Unit vector: \( \hat{\mathbf{v}} = \frac{1}{5} \langle -3, -4 \rangle = \langle -\frac{3}{5}, \frac{-4}{5} \rangle \)
\[ \sqrt{\left(-\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \frac{5}{5} = 1 \]

**Physics Definition**

A vector is a quantity with both direction and size (magnitude).

In this class, we'll examine 2 types of vectors:

1. **Velocity Vector** - gives speed and direction of an object.

2. **Force Vector** - gives the force imparted on an object and the direction of the force. The magnitude of this vector represents amount of force.

**Example:** An airplane is flying at an airspeed of 500 mph at a heading of 330 degrees. A wind of 70 mph is blowing in the direction of N45\(^\circ\)E. Find the ground speed and direction of the plane.

\[ \vec{V} = \text{plane's velocity vector} \]
\[ \vec{W} = \text{wind's velocity vector} \]
\[ \vec{V} + \vec{W} = \text{plane's actual velocity vector, taking into account the wind} \]
Ex. A 120 pound force keeps an 800 pound object from sliding down an inclined ramp. Find the angle of the ramp.
A motorcycle that weighs 811 pounds is placed on a ramp that is inclined 31.8 degrees. Find the magnitude of the force needed to keep the motorcycle from rolling down the ramp, and find the magnitude of the force that the motorcycle exerts against the ramp.

\[ \sin 31.8^\circ = \frac{\|F_2\|}{811} \]
\[ \|F_2\| = 811 \sin 31.8^\circ \approx 427.4 \text{ lb} \]
\[ \|F_1\| = 811 \cos 31.8^\circ \approx 689.3 \text{ lb} \]

**Dot Product**

Let \( \vec{u} = \langle u_1, u_2 \rangle \) and \( \vec{v} = \langle v_1, v_2 \rangle \)

\[ \text{Dot product} = \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 \]

\[ \text{Ex} \quad \vec{u} = \langle 3, 4 \rangle \quad \vec{v} = \langle -2, 1 \rangle \]

\[ \vec{u} \cdot \vec{v} = 3(-2) + 4(1) = -6 + 4 = -2 \]
\[
\begin{align*}
\text{b) } \| \vec{V} \|^2 &= (\sqrt{4 + 1})^2 = 5 \\
\text{c) } \vec{V} \cdot \vec{V} &= \langle -2, 1 \rangle \cdot \langle -2, 1 \rangle \\
&= 4 + 1 \\
&= 5
\end{align*}
\]

**Notes:**
- \( \| \vec{V} \|^2 = \vec{V} \cdot \vec{V} \)
- \( \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \) (Distribution)
- \( c (\vec{u} + \vec{v}) = c \vec{u} + c \vec{v} \) (Scalar Distribution)

**Angle Between Two Vectors**

\[
\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\| \vec{u} \| \| \vec{v} \|}
\]

**Proof:**

**By law of cosines**

\[
\| \vec{u} - \vec{v} \|^2 = \| \vec{u} \|^2 + \| \vec{v} \|^2 - 2 \| \vec{u} \| \| \vec{v} \| \cos \theta
\]

\[
(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2 \| \vec{u} \| \| \vec{v} \| \cos \theta
\]

\[
\vec{u} \cdot \vec{v} - \| \vec{u} \| \| \vec{v} \| \cos \theta = -\frac{1}{2} \| \vec{u} \| \| \vec{v} \| \cos \theta
\]

\[
-2 \vec{u} \cdot \vec{v} = -\frac{1}{2} \| \vec{u} \| \| \vec{v} \| \langle \cos \theta
\]

\[
\vec{u} \cdot \vec{v} = \| \vec{u} \| \| \vec{v} \| \cos \theta
\]
So, \[ \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \cos \theta \]

**Ex.** Find the angle between 
\[ \mathbf{u} = \langle 1, 2 \rangle \quad \mathbf{v} = \langle -3, 4 \rangle \]

\[ \mathbf{u} \cdot \mathbf{v} = -3 + 8 = 5 \]
\[ \|\mathbf{u}\| = \sqrt{5} \]
\[ \|\mathbf{v}\| = 5 \]

\[ \cos \theta = \frac{15}{5\sqrt{5}} = \frac{1}{\sqrt{5}} \]

\[ \theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \]