**Goal:** To calculate and analyze derivatives in any direction

We want to calculate the derivative of \( z = f(x, y) \) in any direction. Let \( \mathbf{u} = \langle u_1, u_2 \rangle \) be a unit vector point in some compass direction in the plane and let \( \mathbf{e} \parallel \mathbf{u} \).

The directional derivative at \((x_0, y_0)\) is the slope of the tangent line (see left) at \((x_0, y_0, z_0)\).

Let \( \mathbf{u} = \langle u_1, u_2 \rangle \) be the direction vector as described above.

Note that the domain of \( z = f(x, y) \) is restricted to \((x, y)\) points on \( \mathbf{e} \):

\[
\mathbf{e} : \hat{f}(t) = \langle x_0 + u_1 t, y_0 + u_2 t \rangle
\]

becomes restricted domain of \( f \).

So \( z = f(x, y) = f(x_0 + u_1 t, y_0 + u_2 t) \)

\[
\frac{\partial z}{\partial x} = \frac{d}{dt} \bigg|_{t=0} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

\[
\frac{\partial z}{\partial x} \bigg|_{x=x_0, y=y_0} = \frac{\partial f(x, y)}{\partial x} u_1 + \frac{\partial f(x, y)}{\partial y} u_2
\]

more generally, \( \frac{\partial z}{\partial x} = f_x(x, y) u_1 + f_y(x, y) u_2 \)
more generally, \[ D_{\vec{u}} f(x, y) = f_x(x, y)u_1 + f_y(x, y)u_2 \]

works as long as \( f \) is differentiable.

(e) Find \( D_{\vec{u}} f(x, y) \) of \( f(x, y) = \cos^{-1}(xy) \)
a at \((1, 0)\) in the direction of \( \vec{v} = \vec{i} + 5\vec{j} \).

need a unit vector: \( \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} \)
\[ \|\vec{v}\| = \sqrt{1 + 25} \]
\[ = \sqrt{26} \]
\[ \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{26}} \vec{i} + \frac{5}{\sqrt{26}} \vec{j} \]

\[ \vec{u} = \frac{1}{\sqrt{26}} \vec{i} + \frac{5}{\sqrt{26}} \vec{j} \]

\[ f(x, y) = \cos^{-1}(xy) \]
\[ f_x = -\frac{y}{\sqrt{1-x^2y^2}} \quad f_y(x, y) = -\frac{x}{\sqrt{1-x^2y^2}} \]
\[ D_{\vec{u}} f(x, y) = -\frac{y}{\sqrt{1-x^2y^2}} \cdot \frac{1}{\sqrt{26}} + -\frac{x}{\sqrt{1-x^2y^2}} \cdot \frac{5}{\sqrt{26}} \]
\[ D_{\vec{u}} f(1, 0) = 0 - \frac{5}{\sqrt{26}} = \frac{-5}{\sqrt{26}} \]

rate of change of \( f \)
rate of change of $f$ at $(1,0,f(1,0))$ in the direction of $\vec{u}$.

Notes:

1. $D_\vec{u} f(x,y) = \left< f_x(x,y), f_y(x,y) \right> \cdot \vec{u} = \nabla f \cdot \vec{u} = \left< u_1, u_2 \right>$
   - gradient of $f$
   - del $f$
   - $\nabla f$

2. $D_\vec{u} f(x,y) = \nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos \phi$
   - $\|\nabla f\| \cos \phi$
   - angle between $\nabla f$ and $\vec{u}$.

3. Properties of $\nabla f = \left< f_x, f_y \right> = f_x \vec{i} + f_y \vec{j}$
   a) max value of $D_\vec{u} f(x,y)$ is $\|\nabla f\|$ (if increases most rapidly in the direction of $\nabla f$)
   b) min value of $D_\vec{u} f(x,y)$ is $-\|\nabla f\|$ (if decreases most rapidly in the direction of $-\nabla f$)
c) $Df(x,y)$ is orthogonal to the level curves of $f$.

If $\vec{u}$ keeps you on a level curve, then $\nabla f \cdot \vec{u} = 0$.

d) $Df(x,y,z)$ is a space vector and is orthogonal to the level surfaces of $f$.

ex) Find the equation of the tangent plane to the surface $z^2 - 2x^2 - 2y^2 - 5 = 0$ at $(1,-1,3)$.

$\nabla F(1,-1,3) \perp F(1,-1,3) = 0$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$F_x \uparrow \quad F_y \uparrow \quad F_z \uparrow$

$z^2 - 2x^2 - 2y^2 - 5 = 0$
\[ F_x(x, y, z) = -4x \quad \Rightarrow \quad F_x(1, -1, 3) = -4 \]
\[ F_y(x, y, z) = -4y \quad \Rightarrow \quad F_y(1, -1, 3) = 4 \]
\[ F_z(x, y, z) = 2z \quad \Rightarrow \quad F_z(1, -1, 3) = 6 \]
\[-4(x - 1) + 4(y + 1) + 6(z - 3) = 0\]

**Ex** Find the path of steepest descent

\[ \text{level curves of some function } f(x, y) \]
\[ \text{Path forms right angles with level curves} \]

**Ex** Let \( f(x, y) = 2xe^{\frac{y}{x}} \). Find the max value of \( \nabla f(x, y) \) at \((2, 0)\).

\[
\begin{align*}
   f_x(x, y) &= 2e^{\frac{y}{x}} + 2x e^{\frac{y}{x}} \cdot \left(\frac{y}{x^2}\right) = 2e^{\frac{y}{x}} - \frac{2y}{x} e^{\frac{y}{x}} \\
   f_y(x, y) &= 2xe^{\frac{y}{x}} \cdot \frac{1}{x} = 2e^{\frac{y}{x}} \\
   \nabla f(x, y) &= \left(2e^{\frac{y}{x}} - \frac{2y}{x} e^{\frac{y}{x}}\right)i + 2e^{\frac{y}{x}} j \\
   \nabla f(2, 0) &= 2i + 2j \\
   \text{max of } \nabla f(2, 0) &= ||\nabla f(2, 0)|| = \sqrt{2^2 + 2^2} = \sqrt{8}
\end{align*}
\]
\[ \max_{\vec{u}} f(2, 0) = \| \nabla f(2, 0) \| = \sqrt{x^2 + 2^2} = \sqrt{8} \]