Goals:

1. To evaluate a limit, when it exists.
2. To show when a particular limit does not exist.
3. To show whether or not a function is continuous.

For the following discussion, $y = f(x)$ is a function of the type you encountered in Calc I.

Recall from calc I that a function is continuous at $a$ if $x$-values that are closer and closer to $a$ (eventually) yield $f(x)$ values that are closer and closer to $f(a)$.

Similarly, for $\lim_{x \to a} f(x) = L$ to exist, as $x \to a$, $f(x) \to L$, even if $f(a)$ is undefined. The $\epsilon - \delta$ Def will make this idea precise...
The $\varepsilon-\delta$ Definition of Limit (calc I)

Assumptions: $y = f(x)$ is defined on $I = (x_1, x_2)$, an open interval containing $a$ (except possibly at $x = a$).

$$\lim_{{x \to a}} f(x) = L \text{ if for all } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that } |f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$ 

Let's generalize this definition for $z = f(x, y)$. 


Informally, this notation says that as 
\((x, y)\) gets closer and closer to \((a, b)\), 
f\((x, y)\) (eventually) gets closer and closer 
to \(L\), assuming \(L\) exists.

**\(\epsilon-\delta\) Def.**

**Definition:** Let \(f(x, y)\) have domain values arbitrarily 
close to \((a, b)\). The \(\lim_{(x,y)\to(a,b)} f(x, y) = L\) if 
for all \(\epsilon > 0\) there exists a \(\delta > 0\) such 
that \(|f(x, y) - L| < \epsilon\) whenever
The Epsilon-Delta definition says nothing about the value of $f(a,b)$ or if it even exists.

1. $(x,y)$ can approach $(a,b)$ from any direction in the plane. For the limit, $L$, of $f(x,y)$ to exist, $L$ must be the same no matter which path $(x,y)$ takes to $(a,b)$.

2. To prove a limit doesn’t exist, find two paths to $(a,b)$ that give different limit values.

Notes on the Epsilon-Delta Definition of Limit

1. The Epsilon-Delta definition says nothing about the value of $f(a,b)$ or if it even exists.

2. $(x,y)$ can approach $(a,b)$ from any direction in the plane. For the limit, $L$, of $f(x,y)$ to exist, $L$ must be the same no matter which path $(x,y)$ takes to $(a,b)$.

3. To prove a limit doesn’t exist, find two paths to $(a,b)$ that give different limit values.

\[
\lim_{{(x,y) \to (0,0)}} \frac{2xy}{2x^2+y^2}
\]

\[
\text{Path } y = x: \lim_{{(x,y) \to (0,0)}} \frac{2xy}{2x^2+y^2} = \lim_{{(x,x) \to (0,0)}} \frac{2x^2}{3x^2} = \frac{2}{3}, \quad (x,y) \neq (0,0)
\]

\[
\text{Path } y = 0: \lim_{{(x,0) \to (0,0)}} \frac{2x \cdot 0}{2x^2 + 0^2} = \lim_{{(x,0) \to (0,0)}} \frac{0}{2x^2} = 0, \quad x \neq 0
\]

So, two paths to $(0,0)$ yield two different values for the limit, which means the original limit ONE.
More Notes on Limits

1. The proof that \( \lim (x,y) \to (a,b) x = a \) is similar to the last example and is left to the student.

2. It is still true \( \lim (x,y) \to (a,b) c = c \), where \( c \) is a constant.

3. The limit laws hold for multi-variable functions (i.e. the limit of a sum(or product) is the sum (or product) of the limits, the limit of a constant times a function is that constant times the limit of the function, etc). The proofs of these laws are similar to the ones for single variable functions.

4. The Squeeze Theorem holds for multi-variable functions.

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**Proof:** Let \( \varepsilon > 0 \). Note that \( \sqrt{(y-b)^2} \leq \sqrt{(x-a)^2+(y-b)^2} \) and \( \sqrt{(y-b)^2} = |y-b| \). Let \( \delta = \varepsilon \), then whenever \( 0 < \sqrt{(x-a)^2+(y-b)^2} < \delta \),

\[ |y-b| \leq \sqrt{(x-a)^2+(y-b)^2} < \delta = \varepsilon. \]

**Done**
**Notes on Continuity**

1. This definition allows us to evaluate limits of continuous functions using substitution.

2. From the limit laws, sums, multiples, and quotients of continuous functions are continuous.

3. From the last example, we know how to show that \( f(x, y) = x \) and \( g(x, y) = y \) are continuous, which means \( h(x, y) = x^m y^n \) is continuous for \( m = \{0, 1, 2, 3, \ldots \} \), since \( h \) is a product of continuous functions.

   Thus, we can conclude that polynomial and rational functions of two variables are continuous on their domains.

4. Thus, we can conclude that polynomial and rational functions of two variables are continuous on their domains.

**A Composition of Continuous Functions is Continuous**

\[ z = f(x, y) \] is continuous at \((a, b)\)

\[ w = g(t) \] is continuous for range values of \( f \)

\[ g(f(a, b)) \] is continuous at \((a, b)\).
\[ f(x, y) = x^2 + y^2 \] \text{ is a continuous polynomial.}

\[ g(x) = \sqrt{x} \] \text{ square root functions are continuous on their domains (calc I result).}

Therefore, \[ g(f(x, y)) = g(x^2 + y^2) = \sqrt{x^2 + y^2} \]
is continuous on its domain, which is all of \( \mathbb{R}^2 \).

\[ \text{Find the limit, if it exists:} \]

\[ \lim_{(x, y) \to (1, 1)} \frac{x}{\sqrt{x + y}} \]

\[ = \frac{1}{\sqrt{1 + 1}} \]

\[ = \frac{1}{\sqrt{2}} \]

\[ \text{a) } \]

\[ \lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x + y} \]

\[ = \lim_{(x, y) \to (0, 0)} \frac{(x+y)(x-y)}{x+y} \]

\[ = \lim_{(x, y) \to (0, 0)} (x-y), \quad x+y \neq 0 \]

\[ y = -x \text{ not in domain} \]
\[ \begin{align*}
&= \lim_{(x,y) \to (0,0)} f(x,y) = f(0,0) = 0 \\
&= 0
\end{align*} \]

\( \Box \) Let \( f(x,y) = \begin{cases} 
\frac{3 x^2 y^3}{x^2 + y^2} & , \quad (x,y) \neq (0,0) \\
0 & , \quad (x,y) = (0,0)
\end{cases} \).

Show that \( f(x,y) \) is continuous at \( (0,0) \).

\([\text{To show: } \lim_{(x,y) \to (0,0)} f(x,y) = f(0,0) = 0]\)

Convert to polar: Let \( x = r \cos \theta \), \( y = r \sin \theta \). Then for \( r \neq 0 \)...

\[ f(x,y) = \frac{3 x y^2}{x^2 + y^2} = \frac{3 r \cos \theta \cdot r^2 \sin^2 \theta}{r^2} = 3 r \cos \theta \sin^2 \theta \]

Note that as \( (x,y) \to (0,0) \), \( r \to 0 \) \((\frac{5}{3}r \to 0)\) and so \( f(x,y) = \frac{3 x y^2}{x^2 + y^2} = 3 r \cos \theta \sin^2 \theta \to 0 \)
and so \( f(x,y) = \frac{2xy}{x^2+y^2} = 3r \cos \theta \sin^2 \theta \to 0 \) by squeeze theorem.

Thus \( \lim \limits_{(x,y) \to (0,0)} f(x,y) = f(0,0) = 0 \) as needed.

Note that there is a hole in the graph of \( z = \frac{3xy^3}{x^2+y^2} \) at the origin. It is called a removable discontinuity. By defining \( f(x,y) \) as above we "removed" this discontinuity. This works because the limit of \( z = \frac{3xy^3}{x^2+y^2} \) as \( (x,y) \to (0,0) \) exists and is 0.

Note: Recall that \( \lim \limits_{(x,y) \to (0,0)} \frac{2xy}{2x^2+y^2} \text{ DNE} \) (we showed this in an earlier example). Below is the graph of \( f(x,y) = \frac{2xy}{2x^2+y^2} \). It is also not defined at \((0,0)\).
But the discontinuity at (0,0) is called non-removable because the limit doesn't exist at (0,0) and therefore can't be "removed."

Note: For \( w = f(x,y,z) \) \( \exists \) A function of 3 inputs

definitions and properties of limits and continuity all hold for functions with 3 (or more) inputs.