Tangent Planes and Linear Approximations

Goals:
1. To find the tangent plane to a surface at a given point.
2. To calculate the total differential of a multivariable function.
3. To determine whether or not a function is differentiable.

Derivation of the Tangent Plane to a Surface at a Point

Note that the tangent plane to \( z = f(x, y) \) at \((x_0, y_0, z_0)\) has equation of the form \( A(x-x_0) + B(y-y_0) + C(z-z_0) = 0 \)
or \( z-z_0 = a(x-x_0) + b(y-y_0) \).

Purple tangent line has equation \( z-z_0 = a(x-x_0) \) in \(xz\)-plane. The slope is \( a = f_x(x_0, y_0) \).

Similarly, if you hold \( x=x_0 \), you'll get \( b = f_y(x_0, y_0) \).

So the equation of the tangent plane is...
(a) Find the equation of the tangent plane to 
\[ f(x, y) = x^2 - 2xy + y^2 \] 
 at \((1, 2, 1)\).

\[ f_x(x, y) = 2x - 2y \implies f_x(1, 2) = 2 - 4 = -2 \]

\[ f_y(x, y) = -2x + 2y \implies f_y(1, 2) = -2 + 4 = 2 \]

\[ z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \]

\[ z - 1 = -2(x - 1) + 2(y - 2) \]

(b) Find a linearization, \( L(x, y) \), for \( f(x, y) \) at \((1, 2, 1)\).

\[ L(x, y) = -2(x - 1) + 2(y - 2) + 1 \]

\[ = -2x + 2 + 2y - 4 + 1 \]

\[ = -2x + 2y - 1 \]

(c) Use \( L(x, y) \) to estimate \( f(0.95, 2.02) \)

\[ L(0.95, 2.02) = -2(0.95) + 2(2.02) - 1 \]

\[ = 1.14 \]

[Turns out \( f(0.95, 2.02) \approx 1.1449 \)]
Consider the tangent plane equation:

\[ z = f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) + z_0. \]

Let \( x = x_0 + \Delta x \) and \( y = y_0 + \Delta y \) and discuss what the result represents geometrically.

\[ z = f_x(x_0, y_0)((x_0 + \Delta x) - x_0) + f_y(x_0, y_0)((y_0 + \Delta y) - y_0) + z_0 \]

\[ = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + z_0 \]

Definitions

1. Total differential

\[ dz = f_x(x, y) \, dx + f_y(x, y) \, dy \]

Recall from Calc I:

\[ dy = f'(x) \, dx \]

\[ \Delta y \approx dy \quad \text{where} \quad \Delta y = f(x_0 + \Delta x) - f(x_0), \]

if differentiable and \( \Delta x \) "small."
\[ \Delta y = \text{change in height of } y = f(x) \text{ from } x_0 \text{ to } x_0 + \Delta x \]

\[ \Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \]

change in height of \( f \) from \((x_0, y_0)\)
to \((x_0 + \Delta x, y_0 + \Delta y)\)

**Note:** For "well-behaved" functions and for "small" \( \Delta x, \Delta y \), \( \Delta z \cong \Delta z \) when going from inputs \((x_0, y_0) \) to \((x_0 + \Delta x, y_0 + \Delta y)\).

\[ \text{ex} \] Let \( z = \sin(2x + 3y) \). Use \( d\ z \) to estimate \( \Delta z \) from \((-3, 2)\) to \((-2.95, 2.04)\)

\[ d\ z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y \]

\[ f_x(x, y) = 2 \cos(2x + 3y) \quad \rightarrow \quad f_x(3, 2) = 2 \]
\[ f_y(x, y) = 3 \cos(2x + 2y) \quad \rightarrow \quad f_y(-3, 2) = 3 \]

\[ d\ z = 2 \cdot (0.05) + 3 \cdot (0.04) = \boxed{0.22} \]

**Note:** \( \Delta z \cong 0.2182 \)

**Note:** 1. In calc I, differentiability \( \Rightarrow \) continuity.

2. **Problem:** Partialials exist \( \not\Rightarrow \) \( f \) is continuous

\[ \uparrow \text{ see (ex) below} \]
[we want a definition of differentiability for \( z = f(x, y) \)
such that \( f \) differentiable \( \Rightarrow \)
\( f \) is continuous.]

(3) solution: come up with a definition of a differentiable function that ensures
\( \Delta z \approx dz \) for "small" input changes.

Definition: \( z = f(x, y) \) is \underline{differentiable} at \((x_0, y_0)\)
as long as \( \Delta z \) can be written as...
\[
\Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y
\]
\[
dz_{(x_0, y_0)}
\]
where \( \epsilon_1, \epsilon_2 \rightarrow 0 \) as \((\Delta x, \Delta y) \rightarrow (0, 0)\).

Theorem 1: \( z = f(x, y) \) \underline{differentiable} at \((x_0, y_0)\)
\[ \Rightarrow f \text{ is continuous at } (x_0, y_0). \]

Proof: uses the above def to show
\[
\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)
\]
means \( f \) is continuous at \((x_0, y_0)\).

Theorem 2: \( f_x, f_y \) continuous on \( D \), then
\( f \) is differentiable on \( D \).
So, just because the partials exist at a point does not mean that the function is well-behaved at that point.
$f$ behaving badly around input $(0,0)$