Goal:

1. To find the absolute extrema of a two-input variable function.
2. To find local (or relative) extrema using the Second Derivative Test.

**Definitions:**

1. Suppose a function $f$ is defined on a closed bounded region $R$ with $f(x_0, y_0) \leq f(x, y) \leq f(x_1, y_1)$ for all $(x, y)$ in $R$. Then $f(x_0, y_0)$ and $f(x_1, y_1)$ are called the *absolute minimum* and *absolute maximum* of $f$ in $R$.
2. A function of two variables has a local maximum at $(a, b)$ if $f(x, y) \leq f(a, b)$ for all points $(x, y)$ in an open disk with center $(a, b)$.
3. A function of two variables has a local minimum at $(a, b)$ if $f(x, y) \geq f(a, b)$ for all points $(x, y)$ in an open disk with center $(a, b)$.
4. A point $(a, b)$ is called a critical point of $f$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

**Theorem 1:** (Extreme Value Theorem) Suppose $f$ is a continuous function of two variables on a closed bounded region $R$ in the $xy$-plane. Then $f$ takes on both an absolute maximum and absolute minimum in $R$.
Theorem 2: If $f$ has a local maximum or minimum at $(a, b)$ and the first-order partial derivatives of $f$ exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Note:

1. Although a critical point is not necessarily a minimum or maximum, local extrema occur only at critical points.
2. The critical points of a function $f$ can also be defined as the points in the domain of $f$ for which $\nabla f(x, y) = \mathbf{0}$ or is undefined. Filling in $0$ for the partial derivatives in the tangent plane formula from (15.4) gives the equation $z = z_0$, a horizontal plane. Thus, the critical values of a function correspond to the points where the function's tangent plane is horizontal or does not exist.

$$\nabla f = f_x \frac{\partial z}{\partial x} + f_y \frac{\partial z}{\partial y}$$

$$= \left< f_x, f_y \right>$$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - z_0 = 0 \text{ when } \nabla f = 0$$

$$z = z_0 \text{ horizontal tangent plane}$$

**Example:** Let $f(x, y) = 3x^2 + 2y^2 - 4y$. Find the absolute extrema over the region $y = x^2$ and $y = 4$.

The region is closed and bounded by graphs, a closed bounded region. By EVT, there will abs extrema on this region since $f$ is a polynomial (all polys are continuous).

1. Interior c.p.'s
   $$f(x, y) = 3x^2 + 2y^2 - 4y$$
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\[ f_x(x, y) = 6x = 0 \implies x = 0 \]
\[ f_y(x, y) = 4y - y = 0 \implies y = 1 \]

\[ \text{c.p. } (0, 1) \]

2. **Boundary** \( y = 1, \ -\frac{1}{2} \leq x \leq 2 \)
\[ f(x, y) = 3x^2 + 2(1)^2 - 4(1) \]
\[ f(x, y) = 3x^2 + 16 \]
\[ \text{c.p. } (0, 4) \]

Also check \( x = -2, \ x = 2 \)
\( (-2, 4), (2, 4) \)

3. **Boundary**: \( y = x^2 \)
\[ f(x, y) = 3x^2 + 2y^2 - 4y \]
\[ f(x, x^2) = 3x^2 + 2(x^2)^2 - 4x^2 \]
\[ g(x) = f(x, x^2) = 2x^4 - x^2 \]
\[ g(x) = 2x^4 - x^2 \]
\[ g'(x) = 8x^3 - 2x = 0 \]
\[ 2x(4x^2 - 1) = 0 \]
\[ 2x = 0 \text{ or } 4x^2 - 1 = 0 \]
\[ \begin{align*}
  x &= 0 \\
  y &= 0 \\
  \end{align*} \]
\[ \begin{align*}
  x &= \pm \frac{1}{2} \\
  (0, 0) \\
  \end{align*} \]

\( \pm \frac{1}{2}, \frac{1}{4} \)
use a table to find the abs. extrema

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>f(x,y) = 3x^2 + 2y^2 - 4y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2 abs. min</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>2.8 abs. max</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.5</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

Theorem 3: **(Second Derivatives Test)** Suppose the second partials of \( f \) are continuous on an open region containing \((a,b)\) and \( \nabla f(a,b) = \mathbf{0} \) (i.e. \((a,b)\) is a critical point). Let

\[
d = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2
\]

a) If \( d > 0 \) and \( f_{xx}(a,b) > 0 \), then \( f(a,b) \) is a local minimum.

b) If \( d > 0 \) and \( f_{xx}(a,b) < 0 \), then \( f(a,b) \) is a local maximum.

c) If \( d < 0 \), then \( f(a,b) \) is neither a local minimum nor a local maximum.

d) The test is inconclusive if \( d = 0 \).

To show part (b) is true, take 2nd order directional derivative in the direction of a generic \( \mathbf{u} \). Show trace is concave down in an open disk containing \((a,b)\) (for any such \( \mathbf{u} \)). Then \( f(a,b) \) is a local max by calc I and derivative test.

Find the local extrema.
\[ f(x, y) = x + y + \frac{1}{xy}, \quad x, y > 0 \]

\[ f_x(x, y) = 1 - \frac{1}{x^2y} = 0 \]

\[ f_y(x, y) = 1 - \frac{1}{x^2y} = 0 \]

\[ \frac{1}{x^2y} = \frac{1}{xy^2} \]

\[ x^2y = x y^2 \]

\[ x^2y - xy^2 = 0 \]

\[ xy(x - y) = 0 \]

\[ x - y = 0 \]

\[ y = x \]

\[ (1, 1) \quad \text{c.p.} \]

\[ d = f_{xx}(1, 1) f_{yy}(1, 1) - (f_{xy}(1, 1))^2 \]

\[ f_{xx}(x, y) = \left| \begin{array}{c} 1 - \frac{1}{x^2y} \\ \frac{1}{x^2y} \\ \frac{1}{xy^2} \end{array} \right| = 0 \]

\[ f_{yx}(x, y) = \left| \begin{array}{c} 1 - \frac{1}{x^2y} \\ \frac{1}{x^2y} \\ \frac{1}{xy^2} \end{array} \right| = 0 \]

\[ f_{yy}(x, y) = \frac{3}{x^3y} \quad \text{as } x \to 0, y \to 0 \]

\[ f_{xy}(x, y) = \frac{2}{xy^3} \quad \to 2 \]

\[ f_{xx}(1, 1) = \frac{2}{x^3y} \quad \to 2 > 0 \]

\[ f_{yy}(1, 1) = \frac{2}{xy^3} \quad \to 2 \]

\[ f_{xy}(1, 1) = \frac{1}{x^2y^2} \quad \to 1 \]
\[
d = 4 - 1 = 3 > 0
\]
\[
f_{xx}(1,1) = 2 > 0, \text{ which means } f(1,1) = 3 \text{ is a } \underline{\text{local minimum}}
\]
\[
\left[ \text{no local max since } f \to \infty \text{ as } x, y \to \infty \text{ or } x, y \to 0 \right]
\]

\[\text{Ex} \]
Find the dimensions of the box with largest volume if the total surface area is 64 cm².

\[
V = xyz
\]
\[
A = 2xz + 2yz + 2xy = 64
\]
\[
xz + yz + xy = 32
\]
\[
(x + y)z = 32 - xy
\]
\[
z = \frac{32 - xy}{x + y}
\]

\[
V(x, y) = xy \left( \frac{32 - xy}{x + y} \right)
\]

\[
V(x, y) = \frac{32xy - x^2y^2}{x + y}
\]

\[
V_x = \frac{(x + y)(32y - 2xy^2) - (32xy - x^3y^2)\cdot 1}{(x + y)^2}
\]
\[
\frac{32xy - 2x^2y^2 + 32y^3 - 2xy^3 - 32xy + 4x^2y^2}{(x+y)^2}
\]

\[
= \frac{-x^2y^2 + 32y^3 - 2xy^3}{(x+y)^2}
\]

\[
\sqrt{x} = \frac{y^2(-x^2 + 32 - 2xy)}{(x+y)^2} = 0
\]

\[
\sqrt{y} = \frac{x^2(-y^2 + 32 - 2xy)}{(x+y)^2} = 0
\]

\[-x^2 + 32 - 2xy = 0 \Rightarrow -x^2 + 32 - 2x^2 = 0
\]

\[-(-y^2 + 32 - 2xy = 0) \Rightarrow -3x^2 = -32
\]

\[-x^2 = -32
\]

\[x^2 = \frac{32}{3}
\]

\[x = \sqrt{\frac{32}{3}} = \sqrt[3]{3}
\]

\[y = \frac{\sqrt[3]{32}}{\sqrt[3]{\sqrt{3}}} = y
\]

\[z = \frac{32 - \frac{32}{3}}{\sqrt[3]{3} + \sqrt[3]{3}}
\]

\[z = \frac{64}{3} \cdot \frac{\sqrt[3]{3}}{2\sqrt[3]{3}} = \frac{32}{2} \cdot \frac{\sqrt[3]{3}}{\sqrt[3]{2}}
\]

\[z = \frac{32}{2} \cdot \frac{\sqrt[3]{3}}{\sqrt[3]{2}}
\]

\[\frac{\sqrt{32}}{\sqrt[3]{3}} = \frac{32}{2} \cdot \frac{\sqrt[3]{3}}{\sqrt[3]{2}}
\]
\[
\frac{\frac{8}{3}}{3} \cdot \frac{\sqrt{3}}{\sqrt{11}} = \frac{8\sqrt{3}}{9\sqrt{11}}
\]

\[
= \frac{4\sqrt{3}}{3} \cdot \frac{\sqrt{11}}{\sqrt{11}} = \frac{4\sqrt{3}}{3}
\]

\[
x = y = 2
\]

The dimensions are \( \sqrt{\frac{3\pi}{3}} \text{ cm} \times \sqrt{\frac{3\pi}{3}} \text{ cm} \times \sqrt{3\pi} \text{ cm} \).