Goal: To solve constrained optimization problems using Lagrange Multipliers

Let $z = f(x, y)$

only allowed to plug in $(x, y)$ values from a curve in $x, y$ plane

Assume $(x_0, y_0)$ is on $g(x, y) = C$

and it produces a local max on $f$ under restricted domain $g$. Note that both $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are normal to $f$ at $(x_0, y_0)$ (since $g$ is tangent to $f$'s level curve at $(x_0, y_0)$). So

$\nabla f \parallel \nabla g$ at $(x_0, y_0) \implies \nabla f = \lambda \nabla g$ at $(x_0, y_0)$.

This leads to the method of Lagrange Multipliers for finding extrema.

$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$

Note:  
1. $\lambda$ is called a Lagrange multiplier
2. Also holds if $(x_0, y_0)$ produces a minimum
Note: 1. \( \lambda \) is called a Lagrange multiplier.
2. It also holds if \((x_0, y_0)\) produces a minimum value on \(f\).
3. On the constraint \(g\), the extrema are absolute.
4. As always, \(f\) is differentiable, \(g\) is smooth.
5. \(Dg(x_0, y_0) \neq 0\) (It could turn out that \(Df(x_0, y_0) = 0\) when \(\lambda = 0\))

Method

1. Find all values of \(x, y\), and \(\lambda\) such that \(Df = \lambda Dg\)

2. Evaluate \(f\) at pts from 1 to find the abs. extrema of \(f\) on the constraint.

Example: Find the extrema of \(f(x, y) = 2x^2 + x + y^2 - 2\) on \(x^2 + y^2 = 4\).

Restricted domain of \(f\) are \((x, y)\) that lie on the circle \(x^2 + y^2 = 4\).

\[
Df = \begin{pmatrix} 4x + 1 \\ 2y \end{pmatrix}, \quad Dg = \begin{pmatrix} 2x \\ 2y \end{pmatrix}
\]

T. real points
\[ \nabla g = \begin{bmatrix} 2x + 2y \end{bmatrix} \]

1. \( 4x + 1 = 2\lambda x \)
2. \( 2y = 2\lambda y \)
3. \( x^2 + y^2 = 4 \)

solve to critical points

implies

\( 2y - 2\lambda y = 0 \Rightarrow 2y(1 - \lambda) = 0 \)
\( \Rightarrow y = 0 \) or \( \lambda = 1. \)

**case I:** \( y = 0 \)

By (3) \( x^2 = 4 \Rightarrow x = \pm 2 \)

\( (\pm 2, 0) \) \( \text{cp}. \)

**case II:** \( \lambda = 1 \)

By (1) \( 4x + 1 = 2x \Rightarrow 2x = -1 \Rightarrow -\frac{1}{2} \)

By (3) \( \left( -\frac{1}{2} \right)^2 + y^2 = 4 \Rightarrow y^2 = 4 - \frac{1}{4} = \frac{15}{4} \)
\( \Rightarrow y = \pm \frac{\sqrt{15}}{2} \Rightarrow \text{cp. \( \left( -\frac{1}{2}, \pm \frac{\sqrt{15}}{2} \right) \).} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( f(x, y) = 2x^2 + x' + y^2 - 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
<td>\text{Abs min} ( f(x, y) = 8 + 2 ) - 2 = 8</td>
</tr>
<tr>
<td>-\frac{1}{2}</td>
<td>±\frac{\sqrt{15}}{2}</td>
<td>\text{Abs min} ( \frac{7}{4} )</td>
</tr>
</tbody>
</table>

**Note:** This method also works for \( w = f(x, y, z) \)

w/ constraint surface \( g(x, y, z) = 0 \)
Ex) Find the extrema using L.M.

\[ f(x, y, z) = x^2 y^2 z^2; \quad x^2 + y^2 + z^2 = 1 \]

\[ Df = 2xy^2z^2 + 2y^2z^2 + 2x^2yz^2 \]

\[ \nabla g = 2\lambda x + 2\lambda y + 2\lambda z \]

1. \( 2xy^2z^2 = 2\lambda x \rightarrow \lambda = y^2z^2 \)
2. \( 2x^2y^2z^2 = 2\lambda y \rightarrow \lambda = x^2z^2 \)
3. \( 2x^2y^2z^2 = 2\lambda z \rightarrow \lambda = x^2y^2 \)
4. \( x^2 + y^2 + z^2 = 1 \)

Note: When \( x = 0 \) or \( y = 0 \) or \( z = 0 \), \( f = 0 \), which is abs. min.

So assume \( x, y, z \neq 0 \).

From 1, 2, 3,

\[ y^2z^2 = x^2z^2 = x^2y^2 \]

\[ y^2 = x^2 = z^2 \]

From 4
\[ x^2 + y^2 + z^2 = 1 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{3}} \]

\[ \left( \pm \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right), \left( \pm \frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \]
\[ \left( \pm \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \]

Critical Points (c.p.): \( \left( \pm \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \)

\[ f \left( \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) = \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} = \frac{1}{27} \]

Note:
Section 14.7 Extrema of Two Variable Functions Page 5

Note:
If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ lie in the same plane, then $\mathbf{w}$ can be written in terms of $\mathbf{u}$ and $\mathbf{v}$ using scalar multiplication and the triangle law.

\[ \mathbf{w} = c\mathbf{u} + k\mathbf{v} \]

Two Constraints

Suppose $W = f(x,y,z)$ has two constraints,
\[ g(x,y,z) = c \quad \text{and} \quad h(x,y,z) = k \]
The domain of $f$ is on the curve of intersection of $h$ and $g$. If $f(x_0,y_0,z_0)$ is an extremum, then it turns out $\nabla f(x_0,y_0,z_0)$ is in the plane created by $\nabla g(x_0,y_0,z_0)$ and $\nabla h(x_0,y_0,z_0)$.

So, $\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0)$

**Lagrange Multipliers**

exercise

Find the extrema using L.M.

\[ f(x,y,z) = 3x - y - 3z \]

\[ \nabla f = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix} \]

\[ \lambda \nabla g = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

\[ \lambda \nabla g = \begin{bmatrix} x + \lambda \\ y + \lambda \\ z + \lambda \end{bmatrix} \]

\[ \nabla f = \lambda \nabla g + \mu \nabla h \]

\[ \begin{cases} x + y - z = 0 \\ x^2 + 2z^2 = 1 \end{cases} \]
\[ \lambda \sigma_g = \lambda^2 + \lambda - \lambda \lambda \]
\[ \mu \sigma h = (2\mu \lambda)^2 + 4\mu \lambda \]

\[ 3 = \lambda + 2\mu \lambda \rightarrow 3 = -1 + 2\mu \lambda \]
\[ -1 = \lambda \]
\[ -3 = -\lambda + 4\mu \lambda \rightarrow -3 = 1 + 4\mu \lambda \]

\[ x + y - z = 0 \]
\[ x^2 + 2z^2 = 1 \]

From (1) and (2), \( x = \frac{y}{2\mu} \Rightarrow x = \left( \frac{2}{\mu} \right) \)

and \( z = \frac{-y}{4\mu} \Rightarrow z = \left( -\frac{1}{4\mu} \right) \)

From (4) \( \left( \frac{2}{\mu} \right)^2 + 2 \left( -\frac{1}{4\mu} \right)^2 = 1 \Rightarrow \frac{y^2}{\mu^2} + \frac{z^2}{\mu^2} = 1 \)

\[ \Rightarrow \frac{6}{\mu^2} = 1 \Rightarrow \mu^2 = 6 \Rightarrow \mu = \pm \sqrt{6} \]

\[ x = \pm \frac{2}{\sqrt{6}} \quad , \quad z = \pm \frac{1}{\sqrt{6}} \]

\[ x = \frac{2}{\sqrt{6}} \]
\[ \frac{2}{\sqrt{6}} + y + \frac{1}{\sqrt{6}} = 0 \Rightarrow y = \left( \frac{-3}{\sqrt{6}} \right) \]

\[ x = -\frac{2}{\sqrt{6}} \]
\[ -\frac{2}{\sqrt{6}} + y - \frac{1}{\sqrt{6}} = 0 \Rightarrow y = \left( \frac{3}{\sqrt{6}} \right) \]
\[ x = \frac{-3}{\sqrt{6}} \]

\[-\frac{3}{\sqrt{6}} + y - \frac{1}{\sqrt{6}} = 0 \quad \Rightarrow \quad y = \frac{3}{\sqrt{6}}\]

\[ (\frac{3}{\sqrt{6}}, \frac{-3}{\sqrt{6}}, \frac{1}{\sqrt{6}}) \quad (\frac{-3}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}) \quad \text{c.p.} \]

\[ f(\hat{a}) = \frac{2}{\sqrt{6}}, \quad f(\hat{b}) = -\frac{2}{\sqrt{6}} \]

\[(\text{ex}) \quad \text{Find the points on the surface} \quad y^2 = 9 + xz \quad \text{that are \underline{closest} to the origin.} \]

\[ \underline{\text{minimizing}} \quad \text{distance squared} \quad \underline{\text{produces same c.p.'s}} \]

\[ d^2 = f(x,y,z) = x^2 + y^2 + z^2 \]

\[ \nabla f = 2x \hat{i} + 2y \hat{j} + 2z \hat{k} \]

\[ \lambda \nabla g = -\lambda z \hat{i} + 2\lambda y \hat{j} - \lambda x \hat{k} \]

\[ 2x = -\lambda z \]
\[ 2y = 2\lambda y \]
\[ 2z = -\lambda x \]
\[ y^2 = 9 + xz \]

\[ \rightarrow (0, \pm 3, 0) \quad \text{c.p.} \]

\[ y^2 = 9 + xz \]

As \( x, z \to \infty, \ y \to \infty \), there is no farthest point away on the surface (from origin). So \( (0, \pm 3, 0) \)
is the closest point on the surface
\( y^2 = z + x^2 \) to the origin.