Section 15.1: Double Integrals  and Volume

**Goal:** To represent the Volume of a solid using a double integral

Look at the picture below. Suppose we wish to find the volume between a rectangular region $R$ in the $xy$-plane and the surface above it. We can approximate the volume by breaking up $R$ into a grid (or mesh) consisting of smaller rectangles and then summing the volume of the columns associated with the smaller rectangles under the surface.

Let $z = f(x,y) \geq 0$. Suppose we want to find the volume between a rectangular region $R$ in the $xy$-plane and $f$. 

$\int_{x=a}^{b} \int_{y=c}^{d} f(x,y) \, dx \, dy$ 

$V_{ij} = \text{i}^{th} \text{j}^{th} \text{box}$ 

$\Delta x \Delta y$
\[
\text{Total volume} = V_{i,j} \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \Delta x \Delta y
\]

\[
V = \lim_{m,n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \Delta x \Delta y = \iint f(x,y) \, dA
\]

**Note:**

1. In general, \( \iint f(x,y) \, dA \) can be calculated by subtracting the volume below the \( xy \)-plane from volume above \( xy \)-plane.

\[
A_1 = \int_a^b f(x) \, dx
\]

\[
A_2 = \int_a^b f(x) \, dx
\]

\[
\iiint f(x,y) \, dA \text{ exists if } f \text{ is continuous.}
\]
3. a) \( \iint_R (f + g) \, dA = \iint_R f \, dA + \iint_R g \, dA \)

b) \( \iint_R cf \, dA = c \iint_R f \, dA \)

c) \( R = R_1 \cup R_2 \) and \( R_1 \cap R_2 = \emptyset \)

\[ \iint_R f \, dA = \iint_{R_1} f \, dA + \iint_{R_2} f \, dA \]

\[ R = [0, \pi] \times [0, \pi] \]

**Example:** Use a Riemann sum with \( m = n = 2 \) to estimate the volume of \( \iiint_R \sin(x + y) \, dA \) where \( R = [0, \pi] \times [0, \pi] \).

a) Use lower left corners as sample points

\[ \Delta x = \frac{\pi}{2}, \quad \Delta y = \frac{\pi}{2} \]

\[ \Delta A = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4} \]

Sample points: \((0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\)
\[ \Delta x = \frac{\pi}{4}, \quad \Delta y = \frac{\pi}{4} \]
\[ \Delta A = \frac{\pi}{2} \cdot \frac{\pi}{4} = \frac{\pi^2}{8} \]

\[ f(x, y) = \sin(x+y) \]

\[ \iint f(x, y) \, dA \approx \frac{\pi^2}{4} \left( f\left(0, 0\right) + f\left(0, \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}, 0\right) + f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) \right) \]
\[ = \frac{\pi^2}{4} \left( \sin(0) + \sin\left(0 + \frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4} + 0\right) + \sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right) \right) \]
\[ = \frac{\pi^2}{4} \left( 0 + 1 + 1 + 0 \right) \]
\[ = \frac{\pi^2}{2} \]

b) Use midpoints as sample pts.

\[ \rho = \left( \frac{\pi}{4}, \frac{\pi}{4} \right), \quad \left( \frac{\pi}{4}, \frac{3\pi}{4} \right), \quad \left( \frac{3\pi}{4}, \frac{\pi}{4} \right), \quad \left( \frac{3\pi}{4}, \frac{3\pi}{4} \right) \]

\[ \iint \sin(x+y) \, dA = \frac{\pi^2}{4} \left[ \sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4} + \frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) + \sin\left(\frac{3\pi}{4} + \frac{3\pi}{4}\right) \right] \]
\[ = \frac{\pi^2}{4} \left( 1 + 0 + 0 + 1 \right) \]
\[ z = \frac{\pi^2}{4} \left[ 1 + 0 + 0 - 1 \right] \]
\[ = 0 \]

\( \text{Ex} \) Evaluate using geometry

\[ \iiint_{R} (4 - 2y) \, dA \]

\[ \mathcal{R} = [0, 1] \times [0, 1] \]

\[ z = 4 - 2y \]

\[ \text{Volume is } \iiint_{R} (4 - 2y) \, dA \]

\[ \iiint_{R} (4 - 2y) \, dA = V_{\text{Box}} + V_{\text{Wedge}} \]

\[ = \frac{1}{8} + \frac{1}{8} \cdot 1 \cdot 1 \cdot 2 \]

\[ = \frac{2 + 1}{2} \]

\[ = 3 \text{ units}^3 \]

Def: The average value of \( z = f(x, y) \) on \( R \).
\[ f_{\text{ave}} = \frac{1}{\text{Area}(R)} \iint_R f(x,y) \, dA \]