Section 12.4: The Dot Product

Goals: 1. To find the dot product of two vectors  
        2. To find the angle between two vectors  
        3. To find the projection of a vector onto another vector  
        4. To find the work done by a constant force

There are two kinds of products defined between vectors, dot and cross products. The dot product gets its name from the fact that it is represented by a dot (•). An important fact to remember is that the dot product of two vectors is always a scalar (that is, a number, not a vector). We will investigate cross products in the next section.

Definition: The dot product of two vectors is the sum of the product of the vectors' corresponding components. For example, suppose \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \) and \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \), then the dot product is given by \( \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \).

Notes: 1. The dot product is commutative and distributive. 
       2. \( \mathbf{0} \cdot \mathbf{v} = 0 \) 
       3. \( \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \) 
       4. See page 843 for a complete list of properties of the dot product.

Theorem 1: If \( \theta \) is the (smaller) angle between two nonzero vectors \( \mathbf{u} \) and \( \mathbf{v} \), then 

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}
\]

Notes: 1. The last theorem gives us an alternate way of writing the dot product: \( \mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos \theta \) 
       2. From the alternative form of the dot product, we can see that, for two nonzero vectors \( \mathbf{u} \) and \( \mathbf{v} \), \( \mathbf{u} \cdot \mathbf{v} = 0 \) if and only if \( \theta = \frac{\pi}{2} \) (i.e. if and only if \( \theta \) is a right angle). 
       3. Two vectors that meet at right angles are called orthogonal.

Definition: The vectors \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal if \( \mathbf{u} \cdot \mathbf{v} = 0 \)

Vector Projections

Suppose we drag a box with force \( \mathbf{u} \), as pictured below. The effective force in the direction of motion (i.e. in the direction of \( \mathbf{v} \)) is called the projection of \( \mathbf{u} \) onto \( \mathbf{v} \), denoted \( \text{proj}_\mathbf{v} \mathbf{u} \). Note that all we need to do to find \( \text{proj}_\mathbf{v} \mathbf{u} \) geometrically is drop a perpendicular down from the terminal point of \( \mathbf{u} \) to the line that contains \( \mathbf{v} \). So, in the pictures below, \( \mathbf{w}_1 = \text{proj}_\mathbf{v} \mathbf{u} \).
To find \( \text{proj}_v u \) analytically, we use the following theorem.

**Theorem 2:** If \( u \) and \( v \) are nonzero vectors, then

\[
\text{proj}_v u = \left( \frac{u \cdot v}{v^2} \right) v
\]

**Note:** The scalar projection of \( u \) onto \( v \) (denoted \( \text{comp}_v u \)) is given by the following formula:

\[
\text{comp}_v u = \frac{u \cdot v}{v^2}
\]

**Work**

Suppose \( d \) represents the distance between points \( P \) and \( Q \). Earlier in your study of calculus, you may have used the formula \( W = \|F\| \cdot d \) to calculate the work performed by a force with a constant magnitude \( \|F\| \) in moving an object through a distance \( d \). Unfortunately, this formula doesn't hold if the force isn't in the direction of motion. In this case, \( \|F\| \) must be replaced by the magnitude of the effective force in the direction of motion, which is \( \|\text{proj}_{PQ} F\| \).

**Definition:** The work \( W \) done by a constant force \( F \) as its point of application moves along the vector \( \overrightarrow{PQ} \) is given by the following formulas.

1. \( W = \|\text{proj}_{PQ} F\| \cdot \|PQ\| \)
2. \( \|F\| \cdot \|PQ\| \cos \theta \)
3. \( W = F \cdot \overrightarrow{PQ} \)

Where \( \theta \) is the angle between \( F \) and \( \overrightarrow{PQ} \).

**Note:** The standard units of work are the foot-pound and the newton-meter (called a joule).