Section 14.2: The Calculus of Vector-Valued Functions

Goals: 1. To differentiate vector-valued functions
2. To integrate vector-valued functions
3. To find a unit tangent vector

As previously stated, the calculus of vector-valued functions is defined to be equivalent to the calculus of its components. This means that

- The derivative of \( r(t) \) exists if the derivative of each of its components exist.
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Notes: 1. These last two statements indicate that we differentiate and integrate vector-valued functions component-wise.
2. The derivative of a vector-valued function is actually defined as a limit:
   \[
   r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}.
   \]
   But, by applying this definition, we can show that differentiation can be performed component-wise as long as the component functions are differentiable (see page 893).

Recall: A function is not differentiable in a place where its graph has a sharp "corner." In plain English, we would say that on a smooth curve there are no sharp corners or cusps.

Definition: A vector-valued function \( r(t) \) is smooth for all \( t \) in an open interval \( I \) if all of its component functions have continuous derivatives on \( I \) and \( r'(t) \neq 0 \) for any \( t \) in \( I \).

Notes: 1. Most of the differentiation rules for vector-valued functions are similar to the ones you learned in calculus I. See page 895 for a complete listing of these rules.
2. There are three "product rules," one for each kind of product: scalar, dot, and cross product.
3. Suppose \( r(t) \) is a position function of an object moving along a curve in space. Then \( v(t) = r'(t) \) is the object's velocity vector. We'll discuss this idea in detail in section 14.4.

Definition: Suppose \( r(t) \) represents a smooth curve on an open interval. The unit tangent vector is given by

\[
T(t) = \frac{r'(t)}{\|r'(t)\|}
\]