Section 15.1: Functions of Several Variables

Goals: 1. To define functions of several variables numerically, algebraically, and visually
2. To sketch the graph of a function of two variables
3. To sketch level curves for a function of two variables
4. To sketch level surfaces for a function of three variables

Definition: Let \( D \) be a set of ordered pairs having real number coordinates. A function of two variables is a rule that assigns each ordered pair \((x, y)\) in \( D \) to a unique real number \( f(x, y) \) in a set \( R \). The set \( D \) is called the domain of \( f \), and the set \( R \) is called the range, that is \( R = \{ f(x, y) \mid (x, y) \in D \} \).

Note: You can think of this type of function as having a domain that is a subset of the \( xy \)-plane and range a subset of the real number line as the below figure indicates.

Example: A multivariable function from a numerical perspective

The temperature-humidity index \( I \) gives the perceived air temperature as a function of the actual temperature \( T \) and the relative humidity \( h \). So \( I \) can be written as \( I = f(T, h) \).

The following table represents \( I \) on a restricted domain.

<table>
<thead>
<tr>
<th>Actual Temperature (°F)</th>
<th>Relative Humidity (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>20</td>
</tr>
<tr>
<td>80</td>
<td>77</td>
</tr>
<tr>
<td>85</td>
<td>82</td>
</tr>
<tr>
<td>90</td>
<td>87</td>
</tr>
<tr>
<td>95</td>
<td>93</td>
</tr>
<tr>
<td>100</td>
<td>99</td>
</tr>
</tbody>
</table>
a) What is the value of $f(95,70)$? What is its meaning?
b) What is the domain of $f$?
c) For what value of $h$ is $f(90,h) = 100$?
d) What is the meaning of the function $I = f(80,h)$?

**Definition:** The level curves of a function $f$ of two independent variables are curves with equations $f(x, y) = c$, where $c$ is a constant in the range of $f$.

**Notes:**
1. Although the graph of a function often gives a good representation of its general behavior, a sketch of level curves can often be easier to read in terms of specific values of the function and certain behavior.
2. Choose $c$ values that are evenly spaced to ensure an accurate depiction of the rate at which the function changes in height.
3. The more closely spaced the level curves, the steeper the graph.
4. Level curves have the following applications.
   a) topographic maps where $z$ is the height of the terrain.
   b) maps of isotherms where $z$ is the temperature at a particular location.
   c) maps of isobars where $z$ is the barometric pressure.

**Definition:** The level surfaces of a function $f$ of three independent variables are surfaces with equations $f(x, y, z) = c$, where $c$ is a constant in the range of $f$.

**Note:** Level surfaces can be used to visualize a function of three variables. Because its graph is embedded in four-dimensional space, we cannot plot a function of three variables in the traditional sense. We can, however, get a glimpse of "hyperspace" by studying the level surfaces of a four dimensional object given by a function of three variables.