Section 15.2: Iterated Integrals

Goals:
1. To evaluate an iterated integral
2. To use an iterated integral to find the area of a rectangular plane region
3. To evaluate a double integral over a rectangular region using an iterated integral

The first thing we need to learn is how to integrate a multi-variable function with respect to a particular variable.

Example 1: Integrate $\int_{1}^{2} (x^2 + 3y^2)dy$.

Note that the above integral is actually a function of $x$. Next, we integrate the function of $x$ that resulted from the first example.

Example 2: Evaluate $\int_{0}^{3} \left[ \int_{1}^{2} (x^2 + 3y^2)dy \right]dx = \int_{0}^{3} (x^2 + 3y^2)dydx$

Notes:
1. The integral in example 2 is an iterated integral. An iterated integral is a special type of definite integral in which the integrand is also an integral. This means we can apply the properties of definite integrals to evaluate iterated integrals.
2. Iterated integrals are used in many applications, including...
   a) area of a 2D region
   b) volume
   c) average of a function
   d) center of mass and moment of inertia
   e) surface area
3. Much like the definite integral from Calculus I, an iterated integral is a number, not a function.
4. The order of integration does not matter. For instance, reversing the order of integration in example 2 will not affect the value of the integral. In other words

$$\int_{0}^{3} \int_{1}^{2} (x^2 + 3y^2)dydx = \int_{1}^{2} \int_{0}^{3} (x^2 + 3y^2)dydx$$

Iterated integrals can be used to evaluate double integrals, as the next theorem suggests.

Theorem (Fubini): Let $f$ be continuous on the rectangle $[a,b] \times [c,d]$, then

$$\int_{R} f(x,y)dA = \int_{a}^{b} \int_{c}^{d} f(x,y)dydx = \int_{a}^{d} \int_{c}^{b} f(x,y)dxdy$$
Notes:

1. To see why Fubini’s Theorem holds, let’s look at an example where the double integral, \( \iint_R f(x, y) dA \), represents volume. The below picture is a vertical slice of the solid region formed between a non-negative function \( f(x, y) \) and a rectangular region \( R \) in the \( xy \)-plane. The side of the shaded region parallel to the \( yz \)-plane has area \( f(x, y) \Delta y \). Thus the integral \( \int_c^d f(x, y) dy \) represents the area of an entire cross section parallel to the \( yz \)-plane. This means the volume of the entire slice must be \( \left[ \int_c^d f(x, y) dy \right] \Delta x \), and hence the volume of the entire solid region between \( f \) and \( R \) is given by the iterated integral \( \iint_a^b f(x, y) dy dx \). So, \( \iint_R f(x, y) dA = \iint_a^b f(x, y) dy dx \)

2. It turns out Fubini’s Theorem generalizes to many types of relatively well-behaved functions that are not necessarily continuous.