Section 15.4: Double Integrals in Polar Coordinates

Goal: To set-up and evaluate double integrals in polar coordinates

First, let's get some preliminaries out of the way. In this section, we are going to have to consider something called a polar rectangle. A polar rectangle is the region formed by two rays with the same starting point and two arcs whose central angle $\Delta \theta$ is the angle between the rays (shown below).

Now, at the microscopic level, a polar rectangle looks a lot like a Cartesian rectangle. So it makes sense that we should be able to approximate the area of a polar rectangle using a Cartesian rectangle. Note that the arc length formula gives us the length, $r \Delta \theta$, of the shorter sector that makes up the above polar rectangle. Thus, the area of the polar rectangle is approximately $r \Delta r \Delta \theta$. Note that this formula gets more and more accurate as $\Delta r$ and $\Delta \theta$ get small.

In this section, we're going to break up the region of integration into a bunch of polar rectangles and integrate using polar coordinates. Based on the above reasoning, it appears that, for polar coordinates, $dA = r dr d\theta$. In fact, it turns out that as long as $f$ is continuous on the region $R$, we have the following result.

$$\int \int_{R} f(x, y) dA = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

for $\theta_1 \leq \theta \leq \theta_2$ and $g_1(\theta) \leq r \leq g_2(\theta)$ and the $g$ functions are continuous on the relevant domains.