Section 15.6: Triple Integrals and Applications

Goals: 1. To find the volume of a solid using a triple integral
2. To find the center of mass and moments of inertia of a solid region

The way we define a triple integral is analogous to the method we used for a double integral. We start by integrating over a solid box in space (as opposed to a rectangular region in the plane) and then extend the definition to integration over any bounded solid region Q. We partition Q with a network of equivalent boxes, using only the boxes that lie entirely within Q. The volume of any particular sub-box is \( \Delta V = \Delta x \Delta y \Delta z \). Next, if we choose any sample point from any particular box to be of the form \((x_i, y_j, z_k)\), then we can create a triple Riemann sum of the form \( \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k) \Delta V \). Thus we can define the triple integral of \( f \) by taking the limit of this sum as the number of sub-boxes goes to infinity.

Definition: If \( f \) is continuous over a bounded solid region \( Q \), then the triple integral of \( f \) over \( Q \) is defined as

\[
\iiint_Q f(x, y, z) \, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k) \Delta V
\]

Assuming this limit exists. The volume of \( Q \) is given by

\[
\text{Volume of } Q = \iiint_Q dV
\]

Note: As with double integrals, triple integrals can be evaluated using iterated integrals.

Theorem 1 (Fubini): Let \( f \) be continuous on a solid region \( Q \) defined by \( a \leq x \leq b, \ h_1(x) \leq y \leq h_2(x), \ g_1(x, y) \leq z \leq g_2(x, y) \) where \( h_1, h_2, g_1, \) and \( g_2 \) are continuous functions. Then,

\[
\iiint_Q f(x, y, z) \, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) \, dz \, dy \, dx
\]

Notes: 1. The above theorem only lists one of the possible six orders of integration.
2. When finding the limits for a specific order of integration, find the inner limits first. Then project \( Q \) onto the coordinate plane of the outer two variables to get a planar region \( R \) and find the remaining limits of integration using the methods for double integrals.
Let $Q$ be a solid region with density function $\rho(x, y, z)$. Below is a summary of several useful formulas from physics.

1. Mass of a solid:
$$m = \iiint_Q \rho(x, y, z) \, dV$$

2. First moment about $yz$-plane:
$$M_{yz} = \iiint_Q x \rho(x, y, z) \, dV$$

3. First moment about $xz$-plane:
$$M_{xz} = \iiint_Q y \rho(x, y, z) \, dV$$

4. First moment about $xy$-plane:
$$M_{xy} = \iiint_Q z \rho(x, y, z) \, dV$$

5. Center of mass:
$$(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$$

6. Moment of inertia about $x$-axis:
$$I_x = \iiint_Q (y^2 + z^2) \rho(x, y, z) \, dV$$

7. Moment of inertia about $y$-axis:
$$I_y = \iiint_Q (x^2 + z^2) \rho(x, y, z) \, dV$$

8. Moment of inertia about $z$-axis:
$$I_z = \iiint_Q (x^2 + y^2) \rho(x, y, z) \, dV$$

9. $I_x = I_{xz} + I_{xy}$, $I_y = I_{yz} + I_{xy}$, and $I_z = I_{xz} + I_{xy}$ where $I_{xy} = \iiint_Q z^2 \rho(x, y, z) \, dV$, $I_{xz} = \iiint_Q y^2 \rho(x, y, z) \, dV$, and $I_{yz} = \iiint_Q x^2 \rho(x, y, z) \, dV$. Use these formulas when you are required to calculate all three moments.