Section 16.1: Vector Fields

Goals:
1. To sketch a vector field
2. To find gradient vector field of a function

Definition: A vector field is a function that maps a point in two or three-dimensional space to a two or three component vector. So, a field of two-dimensional vectors has the form

\[ \mathbf{F}(x, y) = P(x, y)i + Q(x, y)j \]

On the other hand, a field of three-dimensional vectors has the form

\[ \mathbf{F}(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k \]

Consider, for example, the two-dimensional vector field \( \mathbf{F}(x, y) = -yi + xj \). Note that this function represents an infinite number of vectors. An excellent way to picture this vector field is to graph just a few the vectors that it represents using \((x, y)\) as the initial point of each vector (see the picture below).

Notes:
1. For this particular example \( \mathbf{F}(x, y) \cdot \langle x, y \rangle = 0 \), which means each vector \( \mathbf{F}(x, y) \) is perpendicular to the position vector \( \langle x, y \rangle \).
2. In general, vector fields are used to represent phenomenons in a number of areas. A few are listed below.
   a) In a gravitational field each point gives the direction and magnitude of the gravitational force on a particle.
   b) In a magnetic field, each point gives the direction and magnitude of the magnetic force on a particle.
   c) In Fluid Mechanics, each point in a velocity field gives the velocity and direction of a particle of fluid.
3. Keep in mind that vector fields and vector functions are two different types of functions. For instance, a vector function's components are all functions of one variable. It represents a curve in space. On the other hand a vector from a vector field has components that are functions of two or three variables.

Definition: A vector field \( \mathbf{F} \) is called conservative if it is the gradient of a scalar function. In other words, there exists a function \( f \) such that \( \mathbf{F} = \nabla f \). The function \( f \) is called the potential function for \( \mathbf{F} \).