Section 16.5: Curl and Divergence

Goal: 
1. To calculate the curl and divergence of a vector field

Note: 
Even though it is an abuse of notation, we sometimes think of the differential operator \( \nabla \) as a vector \( \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} \).

Definition: 
The curl of \( \mathbf{F}(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k \) is

\[
\text{curl } \mathbf{F}(x,y,z) = \nabla \times \mathbf{F}(x,y,z) = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)i + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)j + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)k
\]

Theorem 1: 
Suppose \( P, Q, \) and \( R \) have continuous first partial derivatives in an open sphere in space. The vector field given by \( \mathbf{F}(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k \) is conservative if and only if \( \text{curl } \mathbf{F}(x,y,z) = 0 \).

Definition: 
The divergence in a plane of \( \mathbf{F}(x,y) = P(x,y)i + Q(x,y)j \) is given by

\[
\text{div } \mathbf{F}(x,y) = \nabla \cdot \mathbf{F}(x,y) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}
\]

The divergence in space of \( \mathbf{F}(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k \) is given by

\[
\text{div } \mathbf{F}(x,y,z) = \nabla \cdot \mathbf{F}(x,y,z) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\]

If \( \text{div } \mathbf{F} = 0 \), then \( \mathbf{F} \) is called divergence free.

Theorem 2: 
If \( \mathbf{F}(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k \) is a vector field and \( P, Q, \) and \( R \) have continuous second partial derivatives, then \( \text{div}(\text{curl } \mathbf{F}) = 0 \).

Theorem 3: 
Green's Theorem in Vector Form

Let \( \mathbf{F}, C, \) and \( R \) have the same conditions on them as given in Green’s Theorem. Then

\[
\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{R} \text{curl } \mathbf{F} \cdot k \, dA.
\]

Theorem 4: 
Let \( \mathbf{F}, C, \) and \( R \) have the same conditions on them as given in Green’s Theorem. Then

\[
\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_{R} \text{div } \mathbf{F}(x,y) \, dA,
\]

where \( \mathbf{n} \) is the outward pointing unit normal vector to \( C \).