# The Basic Building Blocks of Geometry

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<td>• A</td>
<td>Point</td>
<td>• A</td>
<td>point A</td>
<td>• has no size&lt;br&gt;• has no dimension&lt;br&gt;• indicates a definite location&lt;br&gt;• named with an uppercase letter</td>
<td>• pencil point&lt;br&gt;• corner of a room</td>
</tr>
<tr>
<td>A B</td>
<td>Line</td>
<td>→ AB or ← BA</td>
<td>line AB or BA</td>
<td>• is straight&lt;br&gt;• has no thickness&lt;br&gt;• an infinite set of points that extends in opposite directions&lt;br&gt;• one dimension</td>
<td>• highway without boundaries&lt;br&gt;• hallway without bounds</td>
</tr>
<tr>
<td>l</td>
<td>or line l</td>
<td>→ or ←</td>
<td></td>
<td>• is part of a line&lt;br&gt;• has only one endpoint&lt;br&gt;• an infinite set of points that extends in one direction&lt;br&gt;• one dimension</td>
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<td>A B</td>
<td>Ray</td>
<td>→ AB</td>
<td>ray AB</td>
<td>• is part of a line&lt;br&gt;• has only one endpoint&lt;br&gt;• an infinite set of points that extends in one direction&lt;br&gt;• one dimension</td>
<td>• edge of a ruler&lt;br&gt;• base board</td>
</tr>
<tr>
<td>A B</td>
<td>Line</td>
<td>— AB or BA</td>
<td>segment AB or BA</td>
<td>• is part of a line&lt;br&gt;• has two endpoints&lt;br&gt;• one dimension</td>
<td>• floor without boundaries&lt;br&gt;• surface of a football field without boundaries</td>
</tr>
</tbody>
</table>

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LearningExpress Skill Builders  •  LESSON 1
People often use the term angle in everyday conversations. For example, they talk about camera angles, angles for pool shots and golf shots, and angles for furniture placement. In geometry, an angle is formed by two rays with a common endpoint. The symbol used to indicate an angle is \( \angle \). The two rays are the sides of the angle. The common endpoint is the vertex of the angle. In the figure below, the sides are RD and RY, and the vertex is R.

**Naming Angles**

Similar to you, an angle can be named in different ways. The different ways an angle can be named may be confusing if you do not understand the logic behind the different methods of naming.

If three letters are used to name an angle, then the middle letter always names the vertex. If the angle does not share the vertex point with another angle, then you can name the angle with only the letter of the vertex. If a number is written inside the angle that is not the angle measurement, then you can name the angle by that number. You can name the angle below any one of these names: \( \angle WET, \angle TEW, \angle E, \) or \( \angle 1 \).

**Right Angles**

Angles that make a square corner are called right angles. In drawings, the following symbol is used to indicate a right angle:

**Straight Angles**

Opposite rays are two rays that have the same endpoint and form a line. They form a straight angle. In the following figure, \( \overrightarrow{HD} \) and \( \overrightarrow{HS} \) are opposite rays.
PRACTICE

1. Name the vertex and sides of each angle.

2. Name the vertex and sides of each angle.

3. Name the angle in four different ways.

Use the following figure to answer practice problems 4–7.

4. Name a right angle.

5. Name a straight angle.

6. Name a pair of opposite rays.

7. Why is \( \angle O \) an incorrect name for any of the angles shown?

Use the following figure to answer practice problems 8 and 9.

8. \( \overrightarrow{LM} \) and \( \overrightarrow{NL} \) form a line. Are they opposite rays? Why or why not?

9. If two rays have the same endpoints, do they have to be opposite rays? Why or why not?
**Classifying Angles**

Angles are often classified by their measures. The degree is the most commonly used unit for measuring angles. One full term, or a circle, equals 360°.

**Acute Angles**
An acute angle has a measure between 0° and 90°. Here are two examples of acute angles:

- 45°
- 89°

**Right Angles**
A right angle has a 90° measure. The corner of a piece of paper will fit exactly into a right angle. Here are two examples of right angles:

**Obtuse Angles**
An obtuse angle has a measure between 90° and 180°. Here are two examples of obtuse angles:

- 91°
- 170°

**Straight Angles**
A straight angle has a 180° measure. Below is an example of a straight angle (\( \angle ABC \) is a straight angle):

- 180°

*Reflex Angles - greater than 180°*
PRACTICE

Use the figure below to answer practice problems 10–13.

10. Name three acute angles.
11. Name three obtuse angles.
12. Name two straight angles.
13. Name two right angles.

Complete each statement.

14. An angle with measure 90° is called a(n) ____ angle.
15. An angle with measure 180° is called a(n) ____ angle.
16. An angle with a measure between 0° and 90° is called a(n) ____ angle.
17. An angle with a measure between 90° and 180° is called a(n) ____ angle.

Questions 18–21 list the measurement of an angle. Classify each angle as acute, right, obtuse, or straight.

18. 10°
19. 158°
20. 180°
21. 90°

You may want to use a corner of a piece of paper for questions 22–25. Classify each angle as acute, right, obtuse, or straight.
**Complementary Angles**

Two angles are complementary angles if and only if the sum of their measure is $90^\circ$. Each angle is a complement of the other. A pair of angles do not need to be adjacent to be complementary.

In the figure above, $\angle AOB$ and $\angle BOC$ are a pair of complementary angles. $\angle AOB$ and $\angle D$ are also a pair of complementary angles. $\angle AOB$ is a complement of $\angle BOC$ and $\angle D$.

**Supplementary Angles**

Two angles are supplementary angles if and only if the sum of their measures is $180^\circ$. Each angle is a supplement of the other. Similar to complementary angles, supplementary angles do not need to be adjacent.

In the figure above, $\angle XOY$ and $\angle YOZ$ are supplementary angles. $\angle XOY$ and $\angle W$ are also supplementary angles. $\angle XOY$ is a supplement of $\angle YOZ$ and $\angle W$.

**Practice**

Find the measure of a complement of $\angle 1$ for each of the following measures of $\angle 1$.

Find the measure of a supplement of $\angle 2$ for each of the following measures of $\angle 2$.

1. $m\angle 1 = 44^\circ$  
2. $m\angle 1 = 72^\circ$  
3. $m\angle 1 = 80^\circ$  
4. $m\angle 1 = 25^\circ$  
5. $m\angle 1 = 5^\circ$  
6. $m\angle 2 = 88^\circ$  
7. $m\angle 2 = 130^\circ$  
8. $m\angle 2 = 60^\circ$  
9. $m\angle 2 = 115^\circ$  
10. $m\angle 2 = 1^\circ$
Geometric Terms

Being familiar with the roots and prefixes of geometric names is extremely valuable since they appear so often throughout the study of geometry. Much of this nomenclature comes from the Greek and Latin languages because the Greeks and Romans were the first groups to really excel in the study of geometry.

• poly (Greek): many
• gon (Greek): angle
• hedron (Greek): faces
• lateral (Latin): sides
• equi (Latin): equal
• tri (Latin/Greek): three (3)
• quadri (Latin): four (4)
• tetra (Greek): four (4)
• penta (Greek): five (5)
• hexa (Greek): six (6)
• hepta (Greek): seven (7)
• octa (Greek): eight (8)
• nona (Latin): nine (9)
• deca (Latin): ten (10)
• dodeca (Latin): twelve (12)
• icosa (Latin): twenty (20)
Triangles

Naming Triangles by Side Lengths

**Scalene**
All sides have different lengths.

**Isosceles**
Two sides have the same length.

**Equilateral**
All sides have the same length.

Naming Triangles by Angle Measures

**Acute**
All angles have measures less than 90°.

**Right**
One angle has a measure of 90°.

**Obtuse**
One angle has a measure of more than 90°.

Write the name of each triangle according to the measure of its angles.

1. \(35°\) \(115°\) \(30°\)

2. \(80°\) \(80°\) \(20°\)

3. \(90°\) \(50°\) \(40°\)

4. \(15°\) \(150°\) \(15°\)

5. \(12\) cm \(12\) cm \(12\) cm

6. \(18\) cm \(18\) cm \(16\) cm

7. \(9\) cm \(18\) cm \(16\) cm

8. \(3\) cm \(7\) cm \(7\) cm

Write the name of each triangle according to the length of its sides.

Find the degree measure \(x\) in each triangle.

9. \(95°\) \(50°\) \(x\)
\[95° + 50° + x = 180°\]

10. \(90°\) \(48°\) \(x\)

11. \(36°\) \(112°\) \(x\)
Activity

CLASSIFYING TRIANGLES

For each triangle on the Triangles page (p. 141), find the measure of each angle and the length of each side. Indicate the measures inside each figure, then cut out each triangle.

Triangles may be classified in two ways:

A. By the number of congruent sides:
   1. A **scalene** triangle has no congruent sides.
   2. An **isosceles** triangle has at least two congruent sides.
   3. An **equilateral** triangle has three congruent sides.

B. By the types of angles:
   1. An **acute** triangle has three acute angles.
   2. A **right** triangle has a right angle.
   3. An **obtuse** triangle has an obtuse angle.

Sort the triangles using the definitions above. In the blanks below, write the letter of each triangle that belongs in each group. A triangle may belong to more than one group.

1. Scalene
   
2. Acute
   
3. Right
   
4. Equilateral
   
5. Isosceles
   
6. Obtuse
   
7. Is it possible to sort all the triangles using:
   
   a. two classifications by sides (such as isosceles and scalene)? ______
   Explain.

   b. two classifications by angles (such as right and obtuse)? ______
   Explain.

8. List all the possible ways to sort triangles using only two classifications, one by side and one by angle. Sort the triangles according to your groups of two classifications, and indicate the letters of the triangles that belong in each category.

9. Are there any combinations of two classifications, one by side and one by angle, that are not possible? If so, which one(s) and why?
Activity

CLASSIFYING TRIANGLES

For each triangle on the Triangles page (p. 141), find the measure of each angle and the length of each side. Indicate the measures inside each figure, then cut out each triangle.

Triangles may be classified in two ways:

A. By the number of congruent sides:
   1. A scalene triangle has no congruent sides.
   2. An isosceles triangle has at least two congruent sides.
   3. An equilateral triangle has three congruent sides.

B. By the types of angles:
   1. An acute triangle has three acute angles.
   2. A right triangle has a right angle.
   3. An obtuse triangle has an obtuse angle.

Sort the triangles using the definitions above. In the blanks below, write the letter of each triangle that belongs in each group. A triangle may belong to more than one group.

1. Scalene
   B   F   G   I   L   M
   2. Acute
   D   E   H   I
   3. Right
   A   F   G   K
   4. Equilateral
   D   H
   5. Isosceles
   A   C   E   J   K
   6. Obtuse
   B   C   M   N

7. Is it possible to sort all the triangles using:
   a. two classifications by sides (such as isosceles and scalene)? ______
      Explain.
      Yes - scalene, isosceles
   b. two classifications by angles (such as right and obtuse)? ______
      Explain.
      No -

8. List all the possible ways to sort triangles using only two classifications, one by side and one by angle. Sort the triangles according to your groups of two classifications, and indicate the letters of the triangles that belong in each category.

9. Are there any combinations of two classifications, one by side and one by angle, that are not possible? If so, which one(s) and why?
Tangrams

OBJECTIVE: To use tangrams to create various mathematical and non-mathematical pictures; Different patterns and visual problem solving will be explored.

The tangram is a geometric puzzle used to form various figures. This is a great way to get students working with shapes. Use the overhead tangrams to show students the tangram puzzle put together. Then, take the puzzle apart, piece by piece, and explain each piece. Show students how the pieces can fit together to form other figures.

Give each student a copy of the puzzle on the bottom of page 56. Have the students cut the tangram puzzle into seven pieces on the solid lines. Using the seven pieces, have the students put them back together to make the square. Below are some activities to use with the tangrams.

1. Students can work individually to create specific pictures or to invent a few of their own.
2. Have two students “race” each other to piece together a shape or object.
3. Students can race to see how many specific shapes they can create in five minutes. (Keep track of ongoing progress.)

Enlarge the shapes below and on pages 54-56 and make a booklet. Have students flip through the booklet and try to make the figures. There are so many great ways to use tangrams. Have fun!

---

Tangrams

1. 
2. 
3. 
4.
Tangrams continued

5. 

6. \(\text{fish}\) 

7. 

8. 

9. 

10. \(\text{E}\) 

11. 

12. 

13. 

14. 

15. 

16. 

17. 

18. 

19. 

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Tangrams continued

20.

21.

22.

23.

24.

25.

26.

27.

28.

29.

30.

31.

32.

33.

34.

35.
Border Patterns

Look for a pattern to write an expression for the number of \( \bullet \)'s in each figure.

Fig 1  Fig 2  Fig 3  Fig 4  Fig 5  Fig 6

(a) Sketch the next 2 figures.

(b) Complete the following table.

<table>
<thead>
<tr>
<th>figure number</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>50</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of ( \bullet )'s</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Write a description of how the \( \bullet \)'s are arranged for the 50th figure. How many \( \bullet \)'s will there be? Include this information in the table.

(d) Write a description of how the \( \bullet \)'s are arranged for the \( n \)th figure.

(e) Write a general expression for the number of \( \bullet \)'s in the \( n \)th figure. Include this information in the table.
Pattern Plusses

Look for a pattern to write an expression for the number of Os in each figure.

Fig 1  Fig 2  Fig 3  Fig 4

(a) Sketch the next 2 figures.

(b) Complete the following table.

<table>
<thead>
<tr>
<th>figure number</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of Os</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Write a description of how the Os are arranged for the 50th figure. How many Os will there be? Include this information in the table.

(d) Write a description of how the Os are arranged for the nth figure.

(e) Write a general expression for the number of Os in the nth figure. Include this information in the table.
U Patterns

Look for a pattern to write an expression for the number of dots in each figure.

Fig 1  Fig 2  Fig 3  Fig 4  Fig 5  Fig 6

(a) Sketch the next 2 figures.

(b) Complete the following table.

<table>
<thead>
<tr>
<th>figure number</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of dots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Write a description of how the dots are arranged for the 50th figure. How many dots will there be? Include this information in the table.

(d) Write a description of how the dots are arranged for the nth figure.

(e) Write a general expression for the number of dots in the nth figure. Include this information in the table.
Patterns: Try Angle?

Look for a pattern to write an expression for the number of +s in each figure.

Fig 1   Fig 2   Fig 3   Fig 4   Fig 5   Fig 6

(a) Sketch the next 2 figures.
(b) Complete the following table.

<table>
<thead>
<tr>
<th>figure number</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of +s</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Write a general expression for the number of +s in the nth figure. Include this information in the table.

(d) Graph the data in the table. Label the axes.

If you connect the points will the shape of the graph be a straight line?
Look for a pattern to write an expression for the number of $\bullet$s in each figure.

(a) Sketch the next 2 figures.

(b) Complete the following table.

<table>
<thead>
<tr>
<th>figure number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>50</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of $\bullet$s</td>
<td>1</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>196</td>
<td></td>
</tr>
</tbody>
</table>

(c) Write a description of how the $\bullet$s are arranged for the 50th figure. How many $\bullet$s will there be? Include this information in the table.

(d) Write a description of how the $\bullet$s are arranged for the $n$th figure.

(e) Write a general expression for the number of $\bullet$s in the $n$th figure. Include this information in the table.

$$4(n-1)$$
Pattern Plusses

Look for a pattern to write an expression for the number of ⋆s in each figure.

Fig 1  Fig 2  Fig 3  Fig 4

(a) Sketch the next 2 figures.

(b) Complete the following table.

<table>
<thead>
<tr>
<th>figure number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>50</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of ⋆s</td>
<td>5</td>
<td>9</td>
<td>13</td>
<td>17</td>
<td>21</td>
<td>25</td>
<td>29</td>
<td>201</td>
<td>4n + 1</td>
</tr>
</tbody>
</table>

(c) Write a description of how the ⋆s are arranged for the 50th figure. How many ⋆s will there be? Include this information in the table.

\[ \frac{10 \cdot 1}{100} \text{ odd number}
\]
\[ \text{double } + \text{ add } 1 \]

(d) Write a description of how the ⋆s are arranged for the \( n \)th figure.

\[ \frac{n + 1}{4n} \]

(e) Write a general expression for the number of ⋆s in the \( n \)th figure. Include this information in the table.
U Patterns

Look for a pattern to write an expression for the number of os in each figure.

(a) Sketch the next 2 figures.

(b) Complete the following table.

<table>
<thead>
<tr>
<th>figure number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>50</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of os</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td>9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Write a description of how the os are arranged for the 50th figure. How many os will there be? Include this information in the table.

50

(d) Write a description of how the os are arranged for the nth figure.

n

(e) Write a general expression for the number of os in the nth figure. Include this information in the table.

2n + 3
Patterns: Try Angle?

Look for a pattern to write an expression for the number of +s in each figure.

Fig 1  Fig 2  Fig 3  Fig 4  Fig 5  Fig 6

(a) Sketch the next 2 figures.

(b) Complete the following table.

<table>
<thead>
<tr>
<th>figure number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>50</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of +s</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>36</td>
<td>2500</td>
<td>n^2</td>
</tr>
</tbody>
</table>

(c) Write a general expression for the number of +s in the n-th figure. Include this information in the table.

(d) Graph the data in the table. Label the axes.

If you connect the points will the shape of the graph be a straight line? NO
Quadrilaterals
  1. Sum of angles is 360°

Trapezoids
  1 & 2. At least one pair of opposite sides parallel (definition)
  3. Two adjacent sides the same length, and so are the other 2 sides (def).
  4. Diagonals are perpendicular; "long" one bisects "short" one
  5. This diagonal bisects the angle

Parallelograms
  1, 2, & 8. Both pairs of opposite sides are parallel (def).
  9. Both pairs of opposite sides are equal in length.
  10. Either diagonal gives two congruent triangles.
  11. 

Isosceles trapezoids
  1, 2, 6, 7, 7.5, 8, 9, 10, 11, and 12. Has a right angle (def) [actually all 4].

Rectangles
  1, 2, 6, 7, 7.5, 8, 9, 10, 11, and 12. Has a right angle (def) [actually all 4].

Squares
  1, 2, 3, 4, 5, 5.5, 6, 7, 7.5, 8, 9, 10, 11, 12, 13, & 14. 1.5. A square is a rectangle which is a rhombus, or a square is a rectangle with all sides equal.

Kites
  1 & 3. Two adjacent sides the same length, and so are the other 2 sides (def).
  5. This diagonal bisects the angle

Rhombuses
  1, 2, 3, 4, 5, 8, 9, 10, 11, & 13. All sides have the same length.
  14.
2. Decide whether each is always true, sometimes true, or never true.
   a. A square is a rectangle.  \( \text{A} \)
   b. A rectangle is a square.  \( \text{S} \)
   c. A parallelogram is a rectangle.  \( \text{S} \)
   d. A rectangle is a parallelogram.  \( \text{A} \)
   e. A trapezoid is a kite.  \( \text{S} \)
   f. A kite is a trapezoid.  \( \text{S} \)
   g. A kite is a rhombus.  \( \text{S} \)
   h. A rhombus is a kite.  \( \text{A} \)
   i. A square is a kite.  \( \text{A} \)
   j. A kite is a square.  \( \text{S} \)
   k. A right triangle is an isosceles triangle.  \( \text{S} \)
   l. An isosceles triangle is a right triangle.  \( \text{S} \)
   m. An acute triangle is a scalene triangle  \( \text{S} \)
   n. A scalene triangle is an acute triangle.  \( \text{S} \)

3. In each part, what characteristics are shared by the shapes?
   Characteristics might include lengths of sides, sizes of angles,
   parallelism, length of diagonals,... What characteristics are different?
   a. parallelograms, rectangles, rhombi, squares
   b. kites, rhombi, squares
   c. quadrilaterals, trapezoids
   d. trapezoids, parallelograms
   e. parallelograms, rhombi

4. Give an example of each, if possible. If it is not possible, explain why.
   a. a kite which is also a rhombus
   b. an obtuse triangle which is also isosceles
   c. a square which is not a parallelogram  \( \text{not poss.} \)
   d. a parallelogram which is not a square  \( \text{not poss.} \)
   e. a rhombus which is not a kite  \( \text{not poss.} \)
   f. a rectangle which is not a parallelogram  \( \text{not poss.} \)
**Luann** by Greg Evans

Why do you have such a hard time with math, Knute?

I dunno. Math, like, bonks around inside my head.

Once, my 5th grade teacher said, "Every square is a rectangle but not every rectangle is a square."

Man, that one's still bonkin' around...

Counselor?
Properties of Parallelograms

Definition: Opposite sides of a parallelogram are parallel.

Theorem: Opposite sides of a parallelogram are congruent.

Theorem: Opposite angles of a parallelogram are congruent.

Theorem: Consecutive angles of a parallelogram are supplementary.

Theorem: Diagonals of a parallelogram bisect each other.

Be aware that diagonals of a parallelogram are not necessarily congruent. Watch out for this, because many students make this common error.
Other Special Properties

There are a few other special properties for the rectangle, rhombus, and square. First, remember that these figures are all parallelograms; therefore, they possess the same properties of any parallelogram. However, because these figures are special parallelograms, they also have additional special properties. Since a square is both a rectangle and a rhombus, a square possesses these same special properties.

Theorem: The diagonals of a rectangle are congruent.
Theorem: The diagonals of a rhombus are perpendicular, and they bisect the angles of the rhombus.

Theorem: The diagonals of a kite are perpendicular and bisect the angles.
PRACTICE
Use the figure below to find each side length and angle measure for questions 11–15.

11. BC
12. DC
13. \(\angle B\)
14. \(\angle A\)
15. \(\angle C\)

Use the figure below to find each side length and angle measure for questions 16–20.

16. SQ
17. OR
18. \(\angle PQR\)
19. \(\angle SPQ\)
20. \(\angle SRQ\)
QUADRILATERALS

PRACTICE
Use the figure below to find the side length and angle measures for questions 21–24.

_____ 21. \(QS\)  
_____ 22. \(OP\)  
_____ 23. \(\angle QSR\)  
_____ 24. \(\angle RPS\)

Use the figure below to find the angle measures for questions 25–30.

_____ 25. \(\angle JKM\)  
_____ 26. \(\angle JML\)  
_____ 27. \(\angle KJM\)  
_____ 28. \(\angle KLM\)  
_____ 29. \(\angle JNK\)  
_____ 30. \(\angle KNL\)
Finding the Measure of Interior Angles

There are two theorems that you can use to find the measure of interior angles of a convex polygon. One theorem works only for triangles. The other theorem works for all convex polygons, including triangles. Let's take a look at the theorem for triangles first.

Theorem: The sum of the interior angles of a triangle is 180°.

To illustrate this, cut a triangle from a piece of paper. Tear off the three angles or points of the triangle. Without overlapping the edges, put the vertex points together. They will form a straight line or straight angle. Remember that a straight angle is 180°; therefore, the three angles of a triangle add up to 180°, as shown in the figures below.

![Diagram of a triangle with angles torn and put together to form a straight angle.]

You can find the sum of the interior angles of a convex polygon if you know how many sides the polygon has. Look at the figures below and see if you can see a pattern.

![Diagram of polygons with angles torn and put together to form straight angles.]

The diagram suggests that polygons can be divided into triangles. Since each triangle has 180°, the number of triangles is multiplied by 180 to get the sum of the interior angles.

![Diagram of polygons divided into triangles.]

Look for a pattern in the number of sides a polygon has and the number of triangles drawn from one vertex point. You will always have two fewer triangles than the number of sides of the polygon. You can write this as a general statement with the letter $n$ representing the number of sides of the polygon.

Theorem: If a convex polygon has $n$ sides, then its angle sum is given by this formula:

$$S = 180(n - 2)$$
Here is a theorem you can use to find the measure of exterior angles of a convex polygon.

**Theorem:** The sum of the exterior angles of a convex polygon is always 360°.

To illustrate this theorem, picture yourself walking along the sides of a polygon. As you reach each vertex point, you will turn the number of degrees in the exterior angle. When you return to your starting point, you will have rotated 360°.

The figure above shows this theorem using a pentagon. Do you see that this would be true for all polygons as stated in the theorem?

**PRACTICE**

Complete the table for convex polygons

<table>
<thead>
<tr>
<th>Number of sides</th>
<th>7</th>
<th>10</th>
<th>13</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interior ( \angle ) sum</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exterior ( \angle ) sum</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The home plate used in baseball and softball has this shape:

As you can see, there are three right angles and two congruent obtuse angles. What are the measures of the two obtuse angles?
Theorems Concerning the Angles and Diagonals of a Polygon

Here we explore several theorems about convex polygons and provide a brief explanation about why the theorems are true. One of the "proofs" is even explored as a homework exercise. After we look at the theorems, we will further explore their use with a few examples.

**Diagonals of a Polygon**

- There are $n - 3$ diagonals coming from one vertex of an $n$-gon.

Starting at one vertex of an $n$-gon, we draw a diagonal to all but three of the $n$ vertices: we do not draw a diagonal to the starting vertex, or to the two vertices directly adjacent to the starting vertex since they make sides. This is illustrated in the hexagon to the right.

**Sum of the Angles in a Polygon**

- The sum (call it $S$) of the angles of a $n$-gon is given by the formula $S = 180^\circ \cdot (n - 2)$.

Drawing all the diagonals from one vertex of a polygon divides the polygon into $n - 2$ triangles. You should draw a few examples to convince yourself of this fact. Since each triangle has $180^\circ$ in it the total number of degrees in the polygon is $180^\circ(n - 2)$.

**Angle Measures of a Regular Polygon**

- In a regular polygon, the measure of each angle can be found by the formula $A = \frac{180^\circ(n - 2)}{n}$.

The previous theorem says that the sum of all the angles is given by $180^\circ(n - 2)$. In a regular $n$-gon there are $n$ congruent angles. So the measure of each angle is $\frac{180^\circ(n - 2)}{n}$.

**The Number of Sides in a Regular Polygon**

- Given that each angle in a regular polygon measures $A$ degrees, the number of sides is given by the formula $n = \frac{360^\circ}{180^\circ - A}$.

This theorem is a direct result of the previous theorem. The algebraic manipulation is saved as a homework exercise.
<table>
<thead>
<tr>
<th>Conjecture</th>
<th>Opposite sides are equal</th>
<th>Opposite angles are equal</th>
<th>Diagonals bisect the angles</th>
<th>Diagonals bisect each other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quadrilaterals</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trapezoids</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Isosceles trapezoids</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Parallelograms</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rectangles</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Squares</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kites</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rhombuses</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Check your conjectures with the hierarchy.

a. Do all your conjectures about trapezoids appear to apply to parallelograms, rhombuses, rectangles, and squares?

b. Do all your conjectures about kites appear to apply to rhombuses and squares?

c. Do all your conjectures about parallelograms appear to apply to rectangles, rhombuses, and squares?

d. Do all your conjectures about rhombuses appear to apply to squares?

e. Do all your conjectures about rectangles appear to apply to squares?
Squares on the Geoboard

The March 1995 issue of Student Math Notes, "STRETCH ... Your Imagination," involved geoboards. It inspired several mathematical explorations. Students in one class decided to look at a previously solved problem in a slightly different way.

The well-known problem asks, "How many squares are there on a checkerboard?" It seems like a simple problem, since a checkerboard is an 8 by 8 arrangement of squares. However, the question does not ask for the number of small squares. That would simply be $8 \times 8 = 64$ squares. There are other squares, though, ranging in size from $2 \times 2$ up through $8 \times 8$. Thus, the number of squares on a checkerboard is significantly greater than 64. The students' new problem became

"How many squares can be formed with geobands on a geoboard the same size as a checkerboard?"

Remembering how the question in "STRETCH ... Your Imagination" was answered, the students knew they had to get organized. Thus, they started with the smallest geoboard, the $2 \times 2$ size. It has four pegs and is shown here.

![Geoboard Diagram]

Obviously, only one square can be formed. It has vertices $A$, $B$, $C$, and $D$. Even though this solution is not interesting in itself, the students knew that it provided an important beginning.

The students felt better equipped to move on to the next larger geoboard, the $3 \times 3$ size. It has nine pegs, as shown. Obviously, more than one square can be formed. Some of them are like the square formed on the $2 \times 2$ geoboard. One of them is square $ABED$. If this square is translated either to the right or down (sliding without any twisting or turning), other identical squares, with edges parallel to those of $ABED$, are formed.

1. What other squares are formed by translating $ABED$ in this way? 

The students saw that there was a larger square with $A$ as a vertex that could be formed. They realized that the square around the entire geoboard had not been considered yet. That square is $ACIG$. $ABED$ and $ACIG$ are the only possible squares having $A$ as a vertex.

2. How many squares are translations of either $ABED$ or $ACIG$? 

The students then considered squares having $B$ as a vertex. From previous work, they knew they had already counted $ABED$ and $BCFE$. They began looking for squares whose edges were not horizontal or vertical, and they found square $BFHD$ as shown.

From this discussion, we can see that there are two types of squares that have $A$ as a vertex. There are four squares altogether that are translations of $ABED$. $ACIG$ is the only one of its type that can be formed. There is also one additional type of square having $B$ as a vertex that has not been counted. $BFHD$ is the only one of this type that can be formed on the $3 \times 3$ geoboard.

3. How many squares altogether can be formed on a $3 \times 3$ geoboard?

The editors wish to thank Bill Kring, 51 Hidden Lake Court, Savannah, GA 31419, for his contribution to this issue of NCTM Student Math Notes.
The next challenge facing the class was to determine the number of squares that can be formed on a 4 x 4 geoboard like this one.

4. What different squares have A as a vertex?

5. How many squares are translations of each type in the previous question? (Include the original square in your count.)

6. What squares (different from those previously found) have B as a vertex?

7. How many squares are translations of each type in the previous question?

8. What squares (different from those previously found) have C as a vertex?

9. How many squares are translations of each type of square in question 8?

10. Are there any squares that have not yet been considered that have D as a vertex? Explain your answer.

Some interesting patterns emerge from this work. Consider the following table that summarizes them.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Square</th>
<th>Number of squares that are translations</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>ABFE</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>ACKI</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>ADPM</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>BGJE</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>BHOI</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>CLNE</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

Armed with this pattern, the class felt better prepared to determine the number of squares that can be formed on a 5 x 5 geoboard like this one.

11. What different squares have A as a vertex?

12. How many squares are translations of each type of square in the previous question?

13. What squares (different from those previously found) have B as a vertex?

14. How many squares are translations of each type of square in the previous question?

15. What squares (different from those previously found) have C as a vertex?

16. How many squares are translations of each type of square in the previous question?

17. What squares (different from those previously found) have D as a vertex?
18. How many squares are translations of each type of square in question 17? ________________

19. How many squares altogether can be formed on a 5 × 5 geoboard? ________________

After this bit of preliminary work, the class was ready to answer the original question: “How many squares can be formed with geobands on a geoboard the same size as a checkerboard?” The pattern for the 5 × 5 geoboard can be extended to the 9 × 9 geoboard. Suppose the pegs across the top row of the geoboard are labeled A, B, C, D, E, F, G, H, and I.

20. How many squares are translations of the squares that have A as a vertex? ________________

21. How many squares (other than those already found) are translations of those having
   a) B as a vertex? ________________
   b) C as a vertex? ________________
   c) D as a vertex? ________________
   d) E as a vertex? ________________
   e) F as a vertex? ________________
   f) G as a vertex? ________________
   g) H as a vertex? ________________

22. How many squares altogether can be formed on a 9 × 9 geoboard? ________________

The class wondered whether there was a pattern to these values they found. Consider the following table, which summarizes their work on this problem.

<table>
<thead>
<tr>
<th>Number of pegs on a side of the geoboard</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of squares that can be formed</td>
<td>1</td>
<td>6</td>
<td>20</td>
<td>50</td>
<td>105</td>
<td>196</td>
<td>336</td>
<td>540</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

23. What formula will determine the number of squares that can be formed if the number of pegs on a side of the geoboard is known? ________________

24. What values will appear in the table under the 10 and the 11? ________________

Knowing that many good problems can be extended to learn more mathematics, the class considered the sizes of geobands they would need for a 9 × 9 geoboard. Geobands are classified by how far they can stretch before they break. If the rubber bands can be stretched over three adjacent pegs in a row or column, then each side of these bands stretches a maximum of two geoboard units. That kind is called a 2-band. Similarly, 3-bands can be stretched a maximum of three geoboard units before breaking. A 2-band is long enough to stretch all the way around a square with an edge length of 1 geoboard unit.

The class also remembered that they had used the Pythagorean theorem in working problems in “STRETCH ... Your Imagination.” It states that—

“In any right triangle, the sum of the squares of the lengths of the legs of the triangle is equal to the square of the length of the hypotenuse.”

For the triangle shown, this becomes $a^2 + b^2 = c^2$ when written in symbols.

For a 9 × 9 geoboard, the class knew that a 16-band would be big enough to make all squares possible. That information was not particularly interesting to them. They wanted to know what geobands would be needed if each square was formed with the smallest geoband possible. Again they started with a simpler problem, a 5 × 5 geoboard like this one.

25. Write down a square that can be formed with a 2-band. ________________

26. Write down a square that can be formed with a 3-band. ________________

27. Write down a square that can be formed with a 4-band. ________________

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28. Continue this procedure, using the smallest geoboard possible for each square. How many geobands of each size would be needed to show all the possible squares at the same time on a 5 x 5 geoboard?

29. What geobands would be needed on a 9 x 9 geoboard if each square was formed with the smallest geoband possible?

Can you ...

- determine the number of rectangles that can be formed with geobands on a 9 x 9 geoboard?
- determine the number of rectangles that can be formed with geobands on an n x n geoboard?
- determine the number of each kind of geoband needed to form all the squares on an n x n geoboard?
- find the areas of the possible squares on an n x n geoboard and the number of each type of square?
- stretch your imagination and extend this to connecting points inside an n x n x n “geocube”?

Did you know that ...

- points in the coordinate plane that are arranged like pegs on a geoboard are called lattice points?
- graphing calculators can help find formulas like those here using a process called regression?

Mathematical Content

- Geometry, combinatorics, discrete mathematics

Answers

1. BCFE, DEHG, EHI
2. 4 + 1 = 5
3. 6
4. ABFE, ACKI, and ADPM
5. There are 9 like ABFE, 4 like ACKI, and 1 like ADPM.
6. BGJE and BHOI.
7. There are 4 like BGJE and 1 like BHOI.
8. CLNE
9. There is 1 like CLNE.
10. No, all squares having D as a vertex have edges that are vertical or horizontal. Those were already considered with those squares having A as a vertex.
11. ABGF, ACMK, ADSP, and AEYU
12. There are 15 like ABGF, 9 like ACMK, 4 like ADSP, and 1 like AEYU
13. BHLF, BIRK, and BJXP
14. There are 9 like BHLF, 4 like BIRK, 1 like BJXP.
15. CNOF and COWK
16. There are 4 like CNOF and 1 like COWK.
17. DTVF
18. There is one like DTVF.
19. \[1^2 + 2^2 + 3^2 + 4^2 + 1^2 + 2^2 + 3^2 + 2^2 + 1^2 = 50\]
20. \[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 = 204\]
21. a) \[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 = 140\]
    b) \[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91\]
    c) \[1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55\]
    d) \[1^2 + 2^2 + 3^2 + 4^2 = 30\]
    e) \[1^2 + 2^2 + 3^2 = 14\]
    f) \[1^2 + 2^2 = 5\]
    g) \[1^2 = 1\]
22. 540
23. Let n be the number of pegs on a side; number of squares = \((n^2 - n^2)/12\).
24. 825 under 10 and 1210 under 11
25. An example is ABGF.
26. An example is BHLF, since \(BH = \sqrt{2} = 1.414\).
27. ACMK
28. 16 require at least a 2-band, 9 require at least a 3-band, 9 require at least a 4-band, 8 require at least a 5-band, 5 require at least a 6-band, 2 require at least a 7-band, and 1 requires at least an 8-band.
29. 64 require at least a 2-band, 49 require at least a 3-band, 49 require at least a 4-band, 36 require at least a 5-band, 36 require at least a 6-band, 20 require at least a 7-band, 41 require at least an 8-band, 32 require at least a 9-band, 34 require at least a 10-band, 30 require at least an 11-band, 11 require at least a 12-band, 10 require at least a 13-band, 4 require at least a 14-band, 2 require at least a 15-band, and 1 requires at least a 16-band.
Mathematical Symbols

\[ \Delta \] angle
\[ \perp \] right \((90^\circ)\) angle
\[ \triangle \] triangle
\[ \parallel \] parallel to
\[ \perp \] perpendicular to
\[ = \] equal to
\[ \neq \] not equal to
\[ \approx \] approximately equal to
\[ \leq \] less than or equal to
\[ \geq \] greater than or equal to
\[ \sim \] similar to
\[ \not\sim \] not similar to
\[ \cong \] congruent to
\[ \not\cong \] not congruent to
\[ \equiv \] equivalent; identical; congruent to
\[ \not\equiv \] not equivalent to
19.1 Review of Polygon Vocabulary

1. a. scalene right triangle
   b. trapezoid
   c. equilateral triangle (or regular triangle)
   d. rhomboidal region
   e. regular pentagon
   f. parallelogram
   g. square
   h. (equiangular) hexagon
   i. equilateral 12-gon, or concave equilateral 12-gon (see Exercise 6).

   Using less-well known prefixes, it is also an equilateral dodecagon.
   j. rhombus

2. (Instructor: Only g, l, and q are impossible. Ask why.)

3. (Instructor only)
   a. Each vertex is on two sides, so the way of counting given will count each vertex twice.
   b. Each side actually makes two angles, one at each end-point; the way of counting given does not take that fact into account.

4. a. Most of them are isosceles right triangles, but there are also a square and a parallelogram.
   b. There are 7 isosceles right triangles, 2 (overlapping) isosceles trapezoids (not counting special ones), 3 non-isosceles trapezoids (not counting special ones), and 1 parallelogram region (not counting special ones).
   c. (Instructor only) The different sizes are
      1/4 (two pieces), 1/8 (three pieces), and 1/16 (two pieces). Rather than trust one's eyes, copying and cutting out the pieces, and comparing them, gives a convincing justification. You might also justify the results by using area formulas.

5. (Instructor only)
   b. Informally, one might say, "A polygon that does not have any dents," or "One that does not pooch in." A common technical definition is "A polygon is convex if, whenever two of its points are joined, the line segment never goes outside the polygonal region." The idea extends to non-polygonal shapes.
   d. A kite is a quadrilateral that has two consecutive sides the same length, with the other two sides also having equal lengths.

7. d. e. f.
8. Polygon: Number of sides, number of angles, length of each side, size of each angle, total of the lengths of the sides (the perimeter), total of the sizes of the angles,...(see Exercise 9). Polygonal region: The above, plus the area of the region.

9. A table helps to see a pattern that may be less obvious if the results are just written, as with 5 vertices, 5 diagonals; 6 vertices, 9 diagonals,... The pattern can often then be extended from one line to the next (for a 12-gon, 54 diagonals; for a 20-gon, 170 diagonals), but without seeing the general result for $n$ vertices. Possible (and mysterious) hint 1: Add a column for twice the number of diagonals—can you relate that column to the column for the number of vertices? A pattern suggests an educated guess, but, as you will see, patterns cannot always be trusted! Hence, now that you have a conjecture, try to reason why it must be true. Better hint, since it gives a general argument: Each vertex in an $n$-gon is joined to all but 3 of the vertices to give $n-3$ diagonals at each vertex, but doing this at each of the $n$ vertices will count each diagonal twice.

(Instructor: For an $n$-gon, there will be $\frac{n(n-3)}{2}$ diagonals.)

10. The sum of the sizes of the angles in each of the triangles is 180°. When you add all of those up, you are also adding the sizes the angles of the quadrilateral.

11. (Instructor: For the 20-gon, $(20-2)180° = 1440°$. In general, the “angle sum” (the sum of the sizes of the angles) will be $(n-2)180°$. Ask whether there is a justification beyond seeing a pattern.)

12. b. $n^2$, $(x + 1)^2$

c. Hint? It may be helpful to add a new column, $2 \times \#$ dots, and see how the entries in that column are related to the number in the first column, or to consider the differences between successive numbers of dots. Once you have a conjecture based on a pattern, see whether you can give an argument that justifies the conjecture. Instructor: Making an $n$-by-$(n+1)$ “rectangle” with two copies of the right-triangle version helps to see the $\frac{n(n+1)}{2}$ result for the $n$th triangular number.

14. (Instructor only)

a. rectangular region
b. triangle
c. triangular region
d. parallelogram region for the lateral faces, polygonal region of some sort for the base
e. triangular regions for the lateral faces, polygonal region for the base
f. rectangle, most commonly
g. hexagon
h. hexagonal region
Organizing Shapes

1. The classification has each type of quadrilateral in a separate category, even though they share many characteristics besides having four sides. (The circles are a common way of showing the categories, and are not to be viewed as quadrilaterals themselves.

2. a. AT  b. ST  c. ST  d. AT e. ST  f. ST  g. ST
   (Instructors only: h. AT  i. ST  j. ST  k. ST  l. ST  m. ST  n. ST)

3. Samples: a. Shared: opposite sides parallel and equal in length; opposite angles the same size,... Different: Possible for angle sizes to differ, lengths of diagonal can differ,...
   d. Shared: have 4 sides, 4 angles, 2 diagonals, angle sum is 360°,... Different:
      Trapezoids have parallel sides, quadrilaterals may not;...

4. Three are not possible. (Instructor: c, e, and f are not possible.)

5. One possibility, using two new categories...

<table>
<thead>
<tr>
<th>(closed shapes)</th>
<th>(not closed shapes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ovals</td>
<td>zig-zags</td>
</tr>
<tr>
<td>ellipses</td>
<td>squiggles</td>
</tr>
<tr>
<td></td>
<td>angles</td>
</tr>
<tr>
<td></td>
<td>parabolas</td>
</tr>
<tr>
<td>circles</td>
<td></td>
</tr>
</tbody>
</table>

6. a. That the parallel sides are horizontal, with one shorter than the other and above it.
   b. Perhaps the student thinks a parallelogram's angles cannot be right angles—that a parallelogram's sides must be "tilted."
   c. Perhaps the student thinks a rectangle must have unequal dimensions.
   (d,e Instructors only)
   d. Perhaps the student thinks that the angles of a rhombus cannot be equal in size.
   e. Perhaps the student thinks that the sides of a kite cannot all be the same length.
2. (Instructor only) "Opposite sides/angles are equal" means both pairs. For the conjectures listed, there are counterexamples for each conjecture for quadrilaterals and trapezoids and kites, for all but the 3rd conjecture for isosceles trapezoids, for the 3rd and 4th for parallelograms, for the 4th for rectangles. All the conjectures are true for squares, and all but the 3rd is true for rhombuses.

3. (Instructor only) In each case, the properties should apply to the more special polygon.

4. Stuck? Measure segments and angles—are there any possible relationships?

   Instructor: a. The segment joining the midpoints is parallel to, and half the length, of the third side. b. The part a results, of course, and the four triangles are "exactly the same shape" (congruent), hence each has one-fourth the area of the large triangle.

5. a. The angles can be placed next to each other so that the two outside sides appear to lie along a straight line, and their sum is therefore 180°.
   b. The new "placements" of the three angles again appear to lie along a straight line. The method does work with obtuse and right triangles, by folding down the vertex with the largest angle size.

6. (Instructor only)
   a. \( x = 72°; y = 142° \)
   b. \( n = 112°; p = 36°; q = 144° \)
   c. \( s = 75°; r = 95° \) (using the fact that the sum of the sizes of the four angles of the quadrilateral is 360°)

7. a. After the last clue, the shape must be a parallelogram.

9. The sum of the angle sizes of a 20-gon is 3240°, and of an n-gon, \((n - 2)180°\).