Learning Exercises for Section 20.1

1. Curricula in the middle grades may use the informal terms—"mirror image," "slide," "flip," and "turn"—for the isometries. Which technical term goes with each informal term?

   mirror image = reflection
   slide = translation
   flip = reflection (in a line)
   turn = rotation

2. a. Try mental manipulation to tell which single type of rigid motion "transforms" shape A to each of the other shapes. It may help to draw a face or some mark on shape A. You may wish to check by tracing a copy of the shape and transforming it.

   ![Shapes A, B, C, D, E, F]

   b. Which of the shapes in part a are congruent? Explain.
      a. B reflection, C translation, D rotation, E rotation, F translation
      b. Each is congruent to shape A, since it is the image of A for a rigid motion.

3. What rigid motion is involved in each of these? Assume that you are looking at movies, to retain the two-dimensional nature of the work in this section.
   a. A train moving along a straight track
   b. The motion of a fan blade
   c. A child sliding down a playground slide
   d. A clock-hand moving
   e. A skateboarder skating in a circular bowl
   f. A doorknob moving
   a. Translation (unless you focus on the wheels only)
   b. Rotation
   c. Translation (if the slide is straight)
   d. Rotation
   e. Rotation
   f. Rotation (either as it turns, or as the door opens)
4. Make free-hand sketches to show these images of the triangle-rectangle shape below:
   a. for a reflection in line m
   b. for a reflection in line n
   c. for a translation 6 cm to the right
   d. for a rotation of 90° clockwise with center C

   ![Diagram](image)

   e. Which of your shapes is congruent to the original one? Explain.

4.

   ![Diagram](image)

   e. All of the images are congruent to the original shape.
5. The term orientation has a technical meaning that differs from everyday usage of the term. One can assign an orientation to a figure in this way: Pick any three non-collinear points on the figure—call them P, Q, R, say. Then the order P-Q-R assigns a clockwise or counter-clockwise orientation to the figure (it may be helpful to think, clock orientation). Different people, or different choices for the points, may assign a different orientation to the same figure. What is of value is knowing how the orientation of a figure is affected by each type of rigid motion. The orientation of the original and the orientation of its image might be the same, or they might be reversed. Make rough sketches or examine earlier ones to tell how the orientations of an original figure and its image are related for each type of rigid motion.

a. translation b. rotation c. reflection

   a. Translations leave the orientation unchanged.
   b. Rotations leave the orientation unchanged.
   c. Reflections change the orientation.

6. Which of the isometries make sense for “moving” a three-dimensional shape?

   all make sense for

   3-D shapes
Learning Exercises for Section 24.1

1. Find the image for the rigid motion indicated, using paper-tracing.
   
a. The image of a shape of your choice, for a translation with this vector:

b. The image of a shape of your choice, for a rotation with a center of your choice and with a 90° clockwise angle.

c. The image of the shape below, for a reflection in the heavy line segment k.

![Diagram of shapes]

d. The image of a shape of your choice, for a rotation with center on the shape and with a 90° counterclockwise angle.

2. Copy and sketch freehand, as accurately as you can, the image of shape G for each of these rigid motions:

a. A translation 5 cm east (label the image H)

b. A 90° rotation clockwise, center at X (label the image I)

c. A reflection in line m (label the image J)

d. Check your freehand drawings.

e. For you, for which type of rigid motion is it most difficult to visualize images?

No #3
Grid paper is commonly used in elementary school to help in locating images for rigid motions.

a. Find the image $S'$ of the pentagon below, for a translation 4 units west.

b. Find the image $S''$ of the pentagon, for a rotation of 180 degrees, center $P$.

c. Find the image $S'''$ of the pentagon, for a reflection in line $m$.

d. Find the image $S''''$ of the pentagon, for a reflection in line $n$.

e. Find the image $S'''''$ of the pentagon, for a translation with vector $v$. 

[Diagram of a pentagon on a grid with lines labeled $m$, $v$, $n$, and a point labeled $P$]
Find and label the image of the quadrilateral for each part.

a. Image A' for a translation with vector v

b. Image A" for a translation with vector w

c. Image A''' for a reflection in line j

d. Image A'" for a reflection in line k (shade it lightly)

e. Image A''" for a rotation of 90° counter-clockwise, center Q
The regularity of isometric dot paper is also helpful in finding images. Find the image of the flag shape CAT for each of these.

a. Translation with vector \( v \)

b. Reflection in line \( m \)

c. Rotation, 120° clockwise, center C (label it Part c)

d. Rotation, 90° counter-clockwise, center T (label it Part d)

e. Rotation, 60° counter-clockwise, center A (label it Part e)
Copy the quadrilateral ABCD on dot paper and sketch its image for a translation 6 spaces east. Label the image A'B'C'D', where A' is the image of A, B' of B, etc. Draw the line segments joining A and A', B and B', C and C', and D and D'. What do you notice? Does this make sense?

Each line segment is 6 spaces long (to the east)
Using the same ABCD from exercise 6, sketch its image for a 90° clockwise rotation, center at vertex C of the quadrilateral. Label the image A'"B"C"D", where A" is the image of A, B" of B, etc.

a. Join each of A, B, and D to its image with a line segment. Are all these line segments the same length? Does this make sense for a rotation?

b. Join each of A, A", B, B", D and D" to the center, C, with a line segment. Which of these line segments are the same length? Does this make sense for a rotation?

c. Examine the angles made by joining a point to the center C and then to the image of the point. Are all these angles the same size? How large is each of these angles? Does this make sense for a rotation?

\[ \text{Diagram of quadrilateral and its image with labeled points.} \]

a. The segments are not all the same length. Points farther from the center have to "travel" farther than do points closer to the center.

b. \[ \overline{AC} \text{ and } \overline{AC}" \text{ have the same length,} \]
as do \( \overline{BC} \) and \( \overline{BC}" \), and \( \overline{DC} \) and \( \overline{DC}" \). This makes sense for a rotation since each point should stay its distance from the center.

c. Each of the angles is 90°. This result makes sense for a rotation, because each point "travels" through the same angle.
9. Examine one of your reflections. Pick out three or four points on the original shape, and join each point to its image with a line segment. Are all these line segments the same length? What does appear to be true? Does this make sense for a reflection?

The segments can have different lengths, but each is perpendicular to the line of reflection.

10. Which of the following vectors are describing the same translation? Explain.

a, d, g all give the same translation.

c, e, f, and j all describe another translation.

11. Which of these rotations will have the same effect on every point of the plane? The center in each case is the same. Explain your decision.

a. Rotation of 90° clockwise
b. Rotation of 90° counterclockwise
c. Rotation of 180° clockwise
d. Rotation of 180° counterclockwise.

c and d will have the same effect on every point of the plane; turning 180° one clock direction will end a shape up in exactly the same place as will 180° in the other clock direction.
Learning Exercises for Section 24.3

1. For each, draw a triangle with a ruler (and not too close to the edge).
   a. Use the key relationships to draw accurately the image of the triangle, for a translation 3.5 cm east. Write down how you used the key relationships. (Hint: Where are the images of the vertices of the triangle?)
   b. Use the key relationships to draw accurately the image of the triangle for a rotation of 90° counterclockwise, with center at your choice of point. Write down how you used the key relationships.
   c. With a new triangle and your choice for line m, use the key relationships to draw accurately the image of the triangle for the reflection in line m. Write down how you used the key relationships.
2. Trace the shapes in each part and draw accurately each of the following. You may find it helpful to label points and their images. Tell how you used the key relationships.

a. The vector for the translation giving the following (does it matter which one is the original and which one is the image?):

   ![Diagram of a vector for translation]

   Yes - The direction of the vector is determined by which figure is the original and which is the image.

b. The line of reflection for the reflection giving the following (does it matter which one is the original and which one is the image?):

   ![Diagram of a line of reflection]

   No

c. The angle, and its clock direction, for the rotation giving the following; measure the angle also (does it matter which shape is the original and which one is the image?):

   ![Diagram of a rotation]

   Yes - Whether rotation is clockwise or counterclockwise.

d. The line of reflection for the reflection of hexagon RSTUVW giving the following:

   ![Diagram of a reflection of a hexagon]

   W=S' V=T' U'=U' S=W' R=R'
3. For each part, trace the original and the image on separate paper. Identify the rigid motion involved, and then describe the rigid motion fully.

a. reflection in line m
b. translation with vector v
c. reflection in line n
d. rotation, center D, angle about 120° clockwise
e. rotation, center E, angle 180°
Notice that in 2.d above, point R is its own image. If the image of a point is the point itself, that point is called a **fixed point**. Tell where *all* of the fixed points in the plane are, if there are any, for the following.

a. The reflection in some line *k*  
   b. The translation, 4 cm north

c. The rotation with center C, angle 55° clockwise (Hint: There is more than 0.)

d. The rotation with center Q, angle 180°

e. The rotation with center M, angle 360°

4. a. Every point on line *k* is a fixed point.
    b. There are no fixed points.
    c-d. There is one fixed point, namely...  
       the center point only
    e. Every point is a fixed point, for the 360° rotation.
5. Copy the figure and grid below.
   a. Show the image of triangular region ABC, for a 90° clockwise rotation with center P. Label it A'B'C'. (The key relationships may be helpful.)
   b. Now find the image of A'B'C' (notice the primes), for a 90° counterclockwise rotation with center Q. Label it A''B''C''.
   c. What single rigid motion will give A''B''C'' as the image of the original ABC? Describe it as completely as possible (that is, if it is a reflection, say, what would be the line of reflection?).
6.  a. If lines m and n are perpendicular, what is the image of line n for a reflection in line m?

   The line n itself

   b. If k is the bisector of \( \angle ABC \), what is the image of \( \overline{BA} \) for a reflection in line k? The image of \( \overline{BC} \)?

   Ray BC ; Ray BA
Learning Exercises for Section 24.4 1-9, 11-13

1. Make measurements on the first example of composition in the
text to see if there is some relationship between line k, line m, and
the 3 cm east translation.

lines k and m are perpendicular
to the direction of translation
(thus parallel to each other)
and are separated by half the
length of the translation vector
(1.5 cm)
(3 cm)

2. Using your choice of original shape, find the composition of the
two rigid motions in each part. What single rigid motion does the
composition seem to be?

a. (translation, 4 cm east) ° (translation, 2.8 cm west) translation 1.2 cm east

b. (translation, 3 cm northeast) ° (translation, 6 cm southwest)
transl 3 cm southwest

c. (reflection in line n) ° (reflection in line m), when lines n and m
are parallel translation of twice distance between line

d. (reflection in line n) ° (reflection in line m), when lines n and m
are not parallel
rotation - center is intersection of lines
rotation of (x + y), center P

e. (rotation, center P, x° clockwise) ° (rotation, center P, y°
clockwise) rotation of (x + y), center P

f. (rotation, center Q, a° clockwise) ° (rotation, center Q, b°
counterclockwise) rotation of (a - b), center Q
direction depends on which is bigger

g. (rotation, center R, 50° clockwise) ° (rotation, center S, 60°
clockwise), where R and S are different points rotation 110°, center?

h. (rotation, center T, 40° clockwise) ° (reflection in line p), where
line p goes through point T reflection across a line that
is tilted 20° clockwise from p - still going

i. (rotation, center T, 40° clockwise) ° (reflection in line q), where
line q does not go through point T
guide reflection is the simplest
to make rigid motion possible
3. a. How does a glide-reflection affect the (clockwise or counterclockwise) orientation of a figure?

A glide-reflection reverses the orientation of the figure.

b. Summarize how the different types of rigid motions affect the orientation of a figure.

Translations and rotations do not change the orientation (reverse it) but reflections and glide-reflections do.

c. Does a glide-reflection give an image that is congruent to the original shape? Explain your thinking.

Yes it is still congruent to the original shape because it is considered a single rigid motion, and each of the motions involves a shape congruent to the original.

4. What two types of rigid motions might each of the following be, using only orientation as a guide?

a. The composition of a translation followed by a rotation
   rotation or translation

b. The composition of a translation followed by two different reflections
   translation (or rotation if lines not parallel)

c. The composition of 4 different reflections in different lines
   translation or rotation

d. The composition of 17 different reflections
   reflection or glide reflection (shape reversed)

e. The composition of an even number of reflections; an odd number
   translation or rotation

f. The composition of 3 different rotations, followed by 7 different translations, followed by 9 different glide-reflections
   reflection or glide reflection
5.  
   a. Copy and find the image of shape U for the glide-reflection given by the translation with vector w and the reflection in line m. Use paper-tracing if you wish.

   ![Diagram](image)

   b. Label the (final) image of P as P' and of Q as Q'. With a ruler draw the line segments joining P and P' and joining Q and Q'. How does the line of reflection seem to be related to these line segments? Check with other points and their images. This is a key relationship for a point and its image for a glide-reflection.

   The line of reflection symmetry pass thru the midpoints of PP' and QQ'

6.  
   a. In finding the composition of two rigid motions, does it matter in which order the rigid motions are done, in general? Explain.

   In general the order in which rigid motions are done can change the final figure - they are not commutative

   b. With the translation and reflection that define a glide-reflection, does it matter in which order the motions are done? Your finding is the reason why glide-reflections are defined in such a particular way.

   In a glide reflection, order does not matter because line of translation and translation vector are parallel.
7. What single type of rigid motion gives each lettered shape as the image of the given one? Explain how you decided.

- **a. rotation**, \(\approx 85^\circ\) clockwise (with the center below the shapes).
- **b. reflection**, in a vertical line (located between the shapes).
- **c. translation**, \(\approx 2.5\) cm southeast.
- **d. reflection**, in a line tilted \(\approx 35^\circ\) clockwise from vertical (located between the shapes).
- **e. glide-reflection**, reflected in a vertical line (located between the shapes), and then translated vertically down.

8. For each image of QRST, describe the rigid motion which gives the given image as indicated.

- From QRST to Q'R'S'T' by a **glide-reflection** (with horizontal line and vector).
- From QRST to Q''R''S''T'' by a **rotation** (of \(90^\circ\) clockwise, with center below).
- From Q'R'S'T' to Q''R''S''T'' by a **glide-reflection** (with line and vector titled \(45^\circ\) ccw).
9. For each part trace the drawing and find two reflections so that their composition will give the same image as the original rigid motion.

a. This translation

b. This rotation

c. Find other pairs of reflecting lines for parts a and b.

9. a. Your two lines of reflection should be perpendicular to the translation vector, and separated by half the vector's length.

b. Your two lines of reflection should intersect at the center of rotation, and the angle they form should be half the size (≈50°) of the angle of rotation (≈100°).

c. There is an infinite number of possibilities for each part, however each pair of reflections will have the same characteristics as described above.
11. Give an argument for this statement: Any rigid motion can be accomplished with at most three reflections.

11. Every rigid motion is either a **reflection**, a **translation**, a **rotation**, or a **glide-reflection**. Every translation or rotation can be achieved by the composition of two reflections (see #9), and every glide-reflection can be achieved by the composition of three reflections (two for the translation, plus one for the reflection).

12. How many reflections, at minimum, might be needed to achieve the same effect as each of these compositions? Explain your reasoning.
   
   a. A composition of 17 reflections in different lines
   
   b. A composition of 24 reflections in different lines

12. a. **One** reflection (if the lines are parallel or give a similar result), otherwise **three** reflections.
   
   b. **Two** reflections.

13. Which rigid motion(s) could be used in describing each of these?

   a. footprints in the sand
   b. one's right hand, in climbing a ladder
   c. one's two hands, in climbing a ladder
   d. turning a microwave dish
   e. tuning in a different radio station
   f. adjusting a thermostat

13. a. glide-reflection
   b. translation
   c. glide-reflection
   d. rotation
   e. rotation or translation, depending on the type of tuner (dial or lever)
   f. rotation or translation, depending on the type of thermostat
Learning Exercises for Section 24.5

1. Describe all the symmetries of each of these. (An arrow means the pattern continues.)

   a.
   b.

   a. Some include: translation of squares, translation of triangles, rotation of squares + triangle, reflections of squares + triangles
   b. Rotations, translations, reflections all apply

2. Examine both statements in each part below. Is each true? Explain your decisions.

   a. If a shape has translation symmetry, then the shape is infinite.

      If a shape is infinite, then the shape has translation symmetry.

   b. If a shape has glide-reflection symmetry, then the shape is infinite.

      If a shape is infinite, then the shape has glide-reflection symmetry.
3. Which types of symmetry does each of these have?
   a. a line segment   b. a ray   c. a line   d. an angle

   a. two reflection & two rotational symmetries (horz & vert)
   b. 1 reflection symmetry (vertical)
   c. reflection, rotation, translation & glide-reflection
   d. 1 reflection symmetry

4. Design a figure that has translation symmetry. Does your figure also have glide-reflection symmetry?

5. Which transformations are symmetries for the tessellations given by the following? What basic shape gives each tessellation?

   translation (up & down / across)
   rotation (90° & 180°)
   reflection
   glide-reflection
6. Here is information about angle sizes in some triangles. Which of the triangles, if any, are similar? Explain how you know.

Triangle a: 65° and 32°  
Triangle b: 65° and 35°

Triangle c: 35° and 80°  
Triangle d: 65° and 83°

Triangle A: 65°, 32°, 83°
Triangle B: 65°, 35°, 80°
Triangle C: 35°, 80°, 65°
Triangle D: 65°, 83°, 32°

Triangle A and D are similar.  
Triangles B and C are similar.

7. Describe the image of a shape, under the composition of a size transformation with scale factor 1 and a rotation of $x^\circ$, both with the same center.

Congruent to original shape
8. Use the definition of congruence from the transformation view to answer these.

   a. If shape 1 is congruent to shape 2, is shape 2 congruent to shape 1? Explain.  
      \textit{yes} - just reverse the rigid motion
   
   b. Is a shape congruent to itself? Explain.  \textit{yes - 360° rotation}
   
   c. If shape 1 is congruent to shape 2 and shape 2 is congruent to shape 3, is shape 1 congruent to shape 3? Explain.
      \textit{yes, composition of 2 rigid motions is a rigid motion}

9. Use the definition of similarity from the transformation view to answer these.

   a. If shape 1 is similar to shape 2, is shape 2 similar to shape 1? Explain.  \textit{yes - reciprocal of size transformation}
   
   b. Is a shape similar to itself? Explain.  \textit{yes - scale factor = 1}
   
   c. If shape 1 is similar to shape 2 and shape 2 is similar to shape 3, is shape 1 similar to shape 3? Explain.
      \textit{yes - new scale factor is product of previous two scale factors}

10. If it is possible, give an example of two triangles such that each pair of matching angles is the same size, but the triangles are not congruent. Must the two triangles be similar? Explain.

    Triangle can be different sizes, but if all angles match they must be similar.