1. a. Give three factors of 25. Can you find more? If so, how? If not, why not?
   b. Give three multiples of 25. Can you find more? If so, how? If not, why not?
      a) \(25 = 5^2\)
      \[\text{Factors: } 1, 5, 25 \text{ only}\]
      b) 25, 50, 75, more can be found by adding (or multiplying) of 25.

2. a. Write an equation that asserts that 25 is a factor of \(k\). How could a rectangular array show this?
   b. Write an equation that asserts that \(m\) is a factor of \(n\).
   c. Write an equation that asserts that \(r\) is a multiple of \(t\).
      a) \(k = 25 \cdot m\) for some whole \(m\)
         \[
         \begin{array}{c}
         k \\
         \hline
         25 \\
         m
         \end{array}
         \]
      b) \(w = m \cdot n\) for some whole \(n\)
      c) \(v = t \cdot x\) for some whole \(x\)

3. a. If 216 is a factor of 2376, what equation must have a whole number solution?
   b. How does one find out whether 144 is a factor of 3456?
      a) \(216 \cdot x = 2376\) (or \(x = 2376 \div 216\))
      b) Solve \(144 \cdot x = 3456\)
         \(x = 3456 \div 144\) to see if it is a whole number.

4. Use the notion of rectangular arrays to assert that 21 is not divisible by 5.
   \[
   \begin{array}{cc}
   5 & 1 = 5 \\
   2 & 10 \\
   3 & 15 \\
   4 & 20 \\
   5 & 25 \\
   \end{array}
   \]
   \(5 \times 4 = 20\) too small
   \(5 \times 5 = 25\) too big
   cannot make 21
5. Explain why these assertions are not quite correct:
   a. “A factor of a number is always less than the number.”
   b. “A multiple of a number is always greater than the number.”

a) A factor of a number could be the number
b) A multiple of a number could be the number

6. a. Give two factors of 506.
   b. Give two multiples of 506.

   \[ 506 = 2 \cdot 253 \]
   \[ 506, 1012 \]

7. True or false? If false, correct the statement.
   a. 13 is a factor of 39. \( T \)
   b. 12 is a factor of 36. \( T \)
   c. 24 is a factor of 36. \( F \) 24 is a factor of 48
   d. 36 is a factor of 12. \( T \)
   e. 36 is a factor of 48. \( F \) 36 is a factor of 12
   f. 16 is a factor of 512. \( T \)
   g. 2 is a factor of 1. \( F \) 2 is a multiple of 062

8. a. Write an equation that asserts that 15 is a multiple of a whole number \( k \).
   b. Write an equation that asserts that a whole number \( m \) is a factor of a whole number \( x \).

   a) \( 15 = k \cdot x \) for some whole numbers \( k \) and \( x \)
   b) \( x = m \cdot n \) for some whole numbers \( x, m \) in

9. a. Suppose that \( k \) is a factor of \( m \) and \( m \) is a factor of \( n \). Is \( k \) a factor of \( n \)? Is \( n \) a multiple of \( k \)? Justify your decisions. \( \text{Yes} \)
   b. Suppose that \( k \) is a factor of both \( m \) and \( n \). Is \( k \) a factor of \( m + n \) also? Justify your decision. \( \text{Yes} \)
   c. Suppose \( k \) is a factor of \( m \) but \( k \) is not a factor of \( n \). Is \( k \) a factor of \( m + n \) also? Justify your decisions. (You may want to try this with numbers first. For example, 3 is a factor of 15, but is not a factor of \( \underline{\phantom{10}} \)) \( \text{No} \)

a) \( m = k \cdot x \) and \( n = m \cdot y \) for some whole \( # \) \( x \) and \( y \)
   So \( n = k \cdot x \cdot y \) by substitution
b) \( m = k \cdot x \) and \( n = k \cdot y \) for whole \( # \) \( x \) \( y \)
   So \( m + n = k \cdot x + k \cdot y = k \cdot (x + y) \)

c) for example, 3 is a factor of 12 but 3 is not a factor of 10 and so 3 is not a factor of \( 10 + 12 = 22 \)
10. You know that the even (whole) numbers are the elements of the set of numbers
0, 2, 4, 6, 8, ..., and that the odd (whole) numbers are the elements of the set of
numbers 1, 3, 5, 7, 9, ....

a. Write a description of the even numbers that uses "2" and the word "factor."
b. Write a description of the even numbers that uses "2" and the word "mul-
tiple."
c. Write a description of odd numbers that uses "2."

a) an even number is a number that has 2 as a factor
b) an even number is a number that is a multiple of 2
c) an odd number is a number that is not a multiple of 2

11. Complete the following addition and multiplication tables for even and odd num-
ers. Can you then make any definite assertions about . . .

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a. the sum of any number of even numbers? even
b. the sum of any number of odd numbers? even or odd
c. the product of any number of even numbers? even
d. the product of any number of odd numbers? odd
e. whether it is possible for an odd number to have an even factor? no
f. whether it is possible for an even number to have an odd factor? yes
g. Is the set of even numbers closed under addition? yes
h. Is the set of odd numbers closed under addition? no
i. Is the set of even numbers closed under multiplication? yes
j. Is the set of odd numbers closed under multiplication? yes

12. Explain why each of these is a prime number: 2, 3, 29, 97.
The each have exactly two factors

13. List all the primes (prime numbers) less than 100. (You can use the Sieve of
Eratosthenes in the activity on the sieve.)

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47,
53, 59, 61, 67, 71, 73, 79, 83, 89, 97
14. Explain why each of these is a composite number: 15, 27, 49, 119.

\[
\begin{align*}
15 &= 3 \cdot 5 \\
27 &= 3^3 \\
49 &= 7^2 \\
119 &= 7 \cdot 17
\end{align*}
\]

Each has more than 2 factors, including 1 and the number and other factors listed.

15. a. Why is 0 neither a prime nor a composite number?
b. Why is 1 neither a prime nor a composite?
c. What is the drawback to the following "definition" of prime numbers: a whole number with only 1 and itself as factors?

a) Zero has every number as a factor so it has more than 2 factors and is not greater than 1 so is not composite.
b) 1 has only one factor and is not greater than 1.
c) Then 1 would be prime since \(1 \times 1 = 1\).

16. Give two factors of each number (there may be more than two):

a. 829  b. 5771  c. 506  d. \(n\) (if \(n > 1\))

\[
\begin{align*}
a) 1 \times 829 & \quad b) 1 \times 5771 & \quad c) 1 \times 506 & \quad d) 1 \times n
\end{align*}
\]

17. Explain why 2 is the only even prime number. (Can you always find a third factor for larger even numbers?)

Every even \# greater than 2 has itself, 1, and 2 as factors, so it will have at least 3 factors.

18. Conjecture: Given two whole numbers, the larger one will have more factors than the smaller one will. Gather more evidence on this conjecture by working with several (4 or 5) pairs of numbers.

The number of factors is not determined by how large the number is - prime numbers only have 2 factors and can be large, but other cases exist also.

\[36 \rightarrow 1, 2, 3, 4, 6, 9, 12, 18, 36\]
\[45 \rightarrow 1, 3, 5, 9, 15, 45\]
19. a. Just above a number line (at least to 50), mark each factor of 24 with a heavy dot and mark each multiple of 6 with a square.

b. Just below the same number line, mark each factor of 18 with a triangle and mark each multiple of 18 with a circle.

c. What are common factors of 18 and 24? What are common multiples of 6 and 18?

a) Common factors of 6 and 18 are: 1, 2, 3, 6

b) Common multiples of 6 and 18 are: 18, 36

20. Explain without much calculation how you know that 2, 3, 5, 7, 11, 13, and 17 are not factors of $n = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 + 1$.

2 is a factor of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$, but 2 is not a factor of 1, so it cannot be a factor of $n$.

Same is true for the rest.

21. 6 is called a perfect number because its factors (other than itself) add up to the number: $1 + 2 + 3 = 6$. What is the next perfect number?

$$1 + 2 + 4 + 7 + 14 = 28$$
1. State the unique factorization theorem. What does it assert about $239,417$?

That $239,417$ is a prime or can be written as a unique product of primes.

2. Find the prime factorization of each of the following, using a factor tree for each.
   
   a. 102  
   b. 1827  
   c. 1584  
   d. 1540  
   e. 121  
   f. 1485

   \[
   \begin{array}{llllll}
   & 2 & 51 & 3 & 609 & 19 & 99 \\
   & & 3 & 203 & 7 & 11 & 11
   \end{array}
   \]

3. Find the prime factorization of each of these numbers, using a factor tree for at least two of them.
   
   a. 5850  
   b. 256  
   c. 2835  
   d. $10^4$  
   e. 17,280  
   f. Does a complete factor tree for a number show all the factors of the number? All the prime factors of the number?

   a) $2 \cdot 3^2 \cdot 5 \cdot 13$  
   b) $2^8$  
   c) $3^4 \cdot 5 \cdot 7$  
   d) $2^4 \cdot 5^4$  
   e) $2^5 \cdot 3^9 \cdot 5$

4. Name three prime factors of each of the following products.
   
   a. $3 \times 7^3 \times 22$  
   b. $27 \times 22$  
   c. $29^3 \times 11^5 \times 2^5$

   a) 3, 7, 11, 2  
   b) 3, 2, 11  
   c) 29, 11, 2

5. What is the difference between prime factor and prime factorization?

A prime factor can be just one factor. Prime factorization gives all the prime factors.
6. Is it possible to find nonzero whole numbers \(m\) and \(n\) such that \(11^m = 13^n\)? Explain.

NO — \(11^m\) can have only 11's as factors and \(13^n\) can have only 13's as factors so since 11 cannot be a factor of 13 and 13 cannot be a factor of 11
\[11^m \neq 13^n\] for any whole \(m, n\)

7. Which cannot be true, for whole numbers \(m\) and \(n\)? Explain why not. For the ones that can be true, give values for \(m\) and \(n\) that make the equation true.

a. \(2^9\cdot 17^3\cdot 67^2 = 2^7\cdot 17^2\cdot 3^4\cdot 67^m\)
   
   yes:
   
   \[m = 2\cdot 67\]

b. \(2^9\cdot 17^3\cdot 67^2 = 2^9\cdot 17^4\cdot m\)

   no — too many 17's

c. \(2^9\cdot 17^3\cdot 67^2 = 2^8\cdot 17^2\cdot n\)

   \[n = 2\cdot 17\cdot 67^2\]

d. \(2^9\cdot 17^3\cdot 67^3 = 2^9\cdot 17^3\cdot 134\cdot m\)

   NO: \(134 = 67\times 2\) too many 2's

e. \(4^m = 8^n\)

   \[4^m = 2^{2m}\]
   and \(8^n = 2^{3n}\)

   so true if \(2^{2m} = 2^{3n}\)
   or \([2m = 3n]\)

   f. \(6^m = 18^n\)

   \[6^m = (2\cdot 3)^m = 2^m\cdot 3^m\]
   \[18^n = (2\cdot 3^2)^n = 2^n\cdot 3^{2n}\]

   so true if \(n = m\) and \(2n = m\) simultaneously

   works only if \(n = m = 1\)

8. Consider \(m = 2^9\cdot 17^3\cdot 67^2\). Without elaborate calculation, tell which of the following could NOT be factors of \(m\). Explain how you know.

a. \(2^8\cdot 7\) b. \(2^{10}\cdot 17^2\cdot 67\) c. \(2^8\cdot 17^2\cdot 67^2\) d. \(34^3\) e. \(134^2\)

   a) no — \(m\) does not have 7 in its prime factorization
   b) no — it has too many two's
   c) yes
   d) \(34^3 = (2\cdot 17)^3 = 2^3\cdot 17^3\) yes
   e) \(134^2 = (2\cdot 67)^2 = 2^2\cdot 67^2\) yes
9. If 35 is a factor of \( n \), give two other factors of \( n \) (besides 1 and \( n \)).

5 and 7 are also factors

10. How many factors does each have?
   
   a. \( 2^5 \)  
   b. \( 2^2 \cdot 3^3 = 108 \)  
   c. \( 45,000 \)  
   d. \( 2^7 \cdot 3^5 \cdot 11 \cdot 13^2 \)  
   e. \( 10^6 \)  
   f. \( 116 \)  
   g. \( 126 \)  

   Explain your reasoning for two of the parts (a)–(g).

   a) 6 factors
   
   b) \( 3 \cdot 4 = 12 \) factors
   
   c) \( 2^3 \cdot 3^2 \cdot 5^4 \) so \( 4 \cdot 3 \cdot 5 = 60 \) factors
   
   d) \( 8 \cdot 6 \cdot 2 \cdot 3 = 288 \) factors
   
   e) \( 2^6 \cdot 5^6 \) so \( 7 \cdot 7 = 49 \) factors
   
   f) 7 factors
   
   g) \( (2^2 \cdot 3)^6 = 2^{12} \cdot 3^6 \) so \( 13 \cdot 7 = 91 \) factors

11. Consider \( 19^4 \times 11^4 \times 2^5 \). Which of the following products of given numbers are factors of this number for some whole number \( n \)? If so, provide a value of \( n \) that makes it true. If not, tell why not.

   a. \( 19^4 \times 11^3 \times 2^5 \times n \)  
   b. \( 19^4 \times 22 \times 2^5 \times n \)  
   c. \( 19^4 \times 11^4 \times 64 \times n \)  
   d. \( 19 \times 11 \times 2 \times n \)

   a) yes - \( n = 11 \)
   
   b) \( 19^4 \times 11^4 \times 2^4 \times n \) no, too many 2's
   
   c) \( 19^4 \times 11^4 \times 2^6 \times n \) no, too many 2's
   
   d) yes - \( n = 19^3 \times 11^3 \times 2^4 \)

12. Consider \( q = 19^4 \times 11^4 \times 2^5 \). Which of the following are multiples of this number? If so, what would you need to multiply this \( q \) by to get the number?

   a. \( 19^4 \times 11^8 \times 2^5 \)  
   b. \( 19^4 \times 22^4 \times 2^5 \times 17 \)  
   c. \( 19^4 \times 11^4 \times 64 \)  
   d. \( (19 \times 11 \times 2)^5 \)

   a) yes - \( q \times 11^4 \)
   
   b) \( 19^4 \times 2^4 \times 11^4 \times 2^5 \times 17 \) yes - \( q \times 2^4 \times 17 \)
   
   c) \( 19^4 \times 11^4 \times 2^6 \) yes - \( q \times 2 \)
   
   d) \( 19^5 \times 11^5 \times 2^5 \) yes - \( q \times 19 \times 11 \)
13. a. How many factors does 64 have? List them.
   
   \begin{align*}
   2^6 \Rightarrow 7 \text{ factors} \\
   2^4 \cdot 3 \Rightarrow 5 \times 2 = 10 \text{ factors}
   \end{align*}

   b. How many factors does 48 have? List them.

   \begin{align*}
   1, 2, 3, 4, 6, 8, 12, 24, 48
   \end{align*}

   c. How many factors does $19^4 \times 11^4 \times 2^5$ have?

   \begin{align*}
   5 \times 5 \times 6 \text{ factors} = 150 \text{ factors}
   \end{align*}

14. If $p$, $q$, and $r$ are different primes, how many factors does each of the following have?

   a. $p^{10}$
   b. $p^m$
   c. $q^n$
   d. $p^m \cdot q^n$
   e. $p^m \cdot q^n \cdot r^s$

   \begin{align*}
   a) \ (10+1) = 11 \text{ factors} \\
   b) \ (m+1) \text{ factors} \\
   c) \ (n+1) \text{ factors} \\
   d) \ (m+1)(n+1) \text{ factors} \\
   e) \ (m+1)(n+1)(s+1) \text{ factors}
   \end{align*}

15. a. Give two numbers, each of which has exactly 60 factors. (The numbers do not have to be in calculated form.)

   \begin{align*}
   60 &= 6 \times 10 \quad \text{or} \quad 3 \times 20 \\
   60 &= 2^5 \times 3^2 \quad \text{or} \quad 2^2 \times 3^1 \\
   60 &= 3^5 \times 11^1 \quad \text{or} \quad 5^2 \times 7^1
   \end{align*}

16. Give one number that has the number 121 as a factor and that also has exactly 24 factors. Is there just one possibility?

   \begin{align*}
   121 &= 11^2 \\
   24 \text{ factors} &= 3 \times 8 \quad \text{(}\text{example)} \\
   \text{number} &= 11^2 \cdot 3^7 \\
   \text{is one possibility} \\
   11^3 \cdot 5^5 \text{ is another}
   \end{align*}
Learning Exercises for Section 11.3

1. Practice the divisibility tests for 2, 3, 4, 5, 6, 8, 9, and 10 on these numbers:
   a. 43056   b. 700010154   c. 9460000000023   d. 71005165

   a) divisible by 2, 3, 4, 6, 8, 9
   b) divisible by 2, 3, 6, 9
   c) divisible by 3
   d) divisible by 5

2. Give a six-digit number such that
   a. 2 and 3 are factors of the number, but 4 and 9 are not.
   b. 3 and 5 are factors of the number, but 10 is not.
   c. 8 and 9 are factors of the number.

   a) must be even, and digits must sum to a multiple of 3 — but not 9 — and the last 2 digits must not be divisible by 4
      310962 is possible

   b) must end in a 5, and sum to a multiple of 3
      223125 is possible

   c) last 3 digits divisible by 8 and digits sum to a multiple of 9
      125,064 is possible

3. What could a be in \( n = 4187a432 \), if 3 is a factor of \( n \)? If 9 is a factor of \( n \)? Give all the single-digit possibilities for \( a \).

   if \( 3 \) a factor
   \[ 4 + 1 + 8 + 7 + 4 + 3 + 2 = 29 \]
   so \( a = 1, a = 4, a = 7 \) are possible

   if \( 9 \) is a factor
   then \( a = 7 \) is the only possible
4. Notice that 2 and 4 are factors of 12, but 2·4, or 8, is not. So a
    divisibility test for 8 that is NOT safe is to use the 2-test and the 4-test.
    Find counterexamples for these plausible-looking, but unreliable,
    divisibility tests.
    a. 12 is a factor of \( n \) if and only if 2 is a factor of \( n \) and 6 is a factor of \( n \).
    b. 18 is a factor of \( n \) if and only if 3 is a factor of \( n \) and 6 is a factor of \( n \).
    c. 24 is a factor of \( n \) if and only if 4 is a factor of \( n \) and 6 is a factor of \( n \).
      a) 18 is divisible by 2 and 6, not 12
      b) 24 is divisible by 3 and 6, not 18
      c) 36 is divisible by 4 and 6, not 24

5. Try these conjectured divisibility tests with 3 or 4 examples each.
    a. 10 is a factor of \( n \) if and only if 2 is a factor of \( n \) and 5 is a factor of \( n \).
    b. 12 is a factor of \( n \) if and only if 3 is a factor of \( n \) and 4 is a factor of \( n \).
    c. 18 is a factor of \( n \) if and only if 2 is a factor of \( n \) and 9 is a factor of \( n \).
    d. 24 is a factor of \( n \) if and only if 3 is a factor of \( n \) and 8 is a factor of \( n \).
      Examine a–g above to see whether you can predict when such test-two-
      factors approaches will work, and when they will not.
      a) yes
      b) yes
      c) yes
      d) yes

These rules work because the two factors chosen are relatively
prime (have no common factors)
6. Using exercise 5, determine whether...
   a. 24 is a factor of 200000000000000000000112.
   b. 24 is a factor of 20000000000000000000001012.
   c. 18 is a factor of 40000000000000000000000221.
   d. 18 is a factor of 400000000000000000000000212.
   e. 45 is a factor of 111000000000000000000002200.

   a) check \( \frac{8}{3} \) and \( \frac{3}{2} \) for 24
      \[ 112 \div 8 = 14 \quad \text{OK} \]
      \[ 2 + 0 + \ldots + 0 + 1 + 1 + 2 = 6 \div 3 = 2 \quad \text{OK} \]
      \[ \text{YES} \]

   b) check \( \frac{8}{3} \) and \( \frac{3}{2} \)
      \[ 012 \div 8 = 1.5 \quad \text{NO} \]

   c) check \( \frac{9}{2} \) and \( \frac{2}{1} \)
      not even \( \text{NO} \)

   d) check \( \frac{9}{2} \) and \( \frac{2}{1} \)
      even \( \text{OK} \)
      \[ 4 + 0 + \ldots + 0 + 2 + 1 + 2 = 9 \div 9 = 1 \quad \text{OK} \]
      \[ \text{YES} \]

   e) check \( \frac{9}{2} \) and \( \frac{5}{1} \)
      ends in 0 \( \text{OK} \)
      \[ 1 + 1 + 1 + 0 + 2 + 2 + 2 + 0 = 9 \quad \text{OK} \]
      \[ \text{YES} \]

7. Using exercise 6, find a 15-digit number that is a multiple of 36; a 15-digit number that is not a multiple of 36.

check \( \frac{9}{4} \) and \( \frac{4}{1} \) last 2 digits a multiple of 4
   if sum divisible by 9

for example
   \[ 117324 \]
   so \[ 117324 + 1 = 117325 \]
   not divisible by 4 or 9
8. The divisibility tests given here depend on the number being expressed in base ten. The tests are properties of the numeration system rather than of the numbers. Find examples with numbers written in base five to show that, say, the (base-ten) divisibility test for two does not work in base five. You will want to find a number for which two is (or is not) a factor but whose base five representation does (or does not) satisfy the divisibility test for two that you know for base ten.

9. Explain why finding only one factor of \( n \) besides 1 and \( n \) is enough to show that \( n \) is composite.

A composite is defined as having more than two factors, so if you can find a 3rd factor, it is composite.

10. Suppose that \( n = 2^3 \times 5^2 \times 7 \times 17^3 \). Give the prime factorization of \( n^2 \) and \( n^3 \). (Hint: Do not work too hard.)

\[
\begin{align*}
n^2 &= (2^3 \times 5^2 \times 7 \times 17^3)^2 \\
&= 2^6 \times 5^4 \times 7^2 \times 17^6 \\
n^3 &= (2^3 \times 5^2 \times 7 \times 17^3)^3 \\
&= 2^9 \times 5^6 \times 7^3 \times 17^9
\end{align*}
\]
11. Determine whether each of these is a prime:

a. 667  
   b. 289  
   c. 3501  
   d. 47 \times 61  
   e. 4319  
   f. 29^3  

   a) \(26^2 = 676\) so check primes less than 26
      \(23 \times 29 = 667\) - not prime
   b) \(17^2 = 289\) not prime
   c) divisible by 9
   d) already has more than 2 factors
   e) \(66^2 = 4316\) so check primes less than 66
      \(7 \times 617 = 4319\) - not prime
   f) 29 and 29^2 are factors - so has more than 2 factors

12. The numbers 2 and 3 are consecutive whole numbers, each of which is a prime.
Is there another pair of consecutive whole numbers, each of which is a prime?
If there is, find such a pair; if not, explain why.

   No - any other consecutive pair of whole numbers contains an even number, divisible by 2, so cannot be possible

13. Test each of these numbers for divisibility by 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, and 18.
   a. 540  
   b. 150  
   c. 145  
   d. 369  
   e. 840  

   a) 2, 3, 4, 5, 6, 9, 10, 12, 15, 18
   b) 2, 3, 5, 6, 10, 15
   c) 5
   d) 3, 9
   e) 2, 3, 4, 5, 6, 8, 10, 12, 15
14. Which of these numbers are prime? For those not prime, give the prime factorization.

- a) not prime  
  P.F. = 2 \times 3 \times 29
- b) not prime  
  P.F. = 2^3 \times 7 \times 107
- c) prime
- d) not prime  
  P.F. = 11^2
- e) not prime  
  P.F. = 31^2
- f) not prime  
  P.F. = 29 \times 43
- g) not prime  
  P.F. = 2^3 \times 3^2 \times 53

15. Here is an interesting conjecture that mathematicians are uncertain about even though it has been studied for more than 100 years: Every even number greater than 2 can be written as the sum of two primes (Goldbach's conjecture). Test the conjecture for the even numbers through 36.

- 4 = 2 + 2
- 6 = 3 + 3
- 8 = 3 + 5
- 10 = 5 + 5
- 12 = 5 + 7
- 14 = 7 + 7
- 16 = 7 + 9
- 18 = 9 + 9
- 20 = 9 + 11
- 22 = 11 + 11
- 24 = 11 + 13
- etc.

16. Devise a way of checking to see whether or not a number is divisible by 24. Test your method on 36; on 120.

Tests must be relatively prime.
Use 8 and 3 to test 24.

17. Are every two different primes relatively prime? Explain.

Yes, they will share no factors.

18. Which pairs of numbers are relatively prime?

- a) yes
- b) no
- c) yes
- d) yes
- e) no
- f) yes
- g) yes
- h) no
- i) yes
- j) no
- k) no
- l) no
- m) no
- n) yes
- o) no
- p) yes
- q) yes
- r) no
- s) yes
- t) no
- u) yes
- v) no
- w) yes
- x) no
- y) yes
- z) no

a. 2, 5  b. 2, 4  c. 2, 6  d. 2, 7  e. 2, 8  f. 2, 9

- g. 8, 9  h. 8, 12  i. 3, 8  j. 40, 42  k. 121, 22  l. 39, 169

- a) yes  b) no  c) no  d) yes  
- e) no  f) yes  g) yes  h) no  i) yes  j) no  k) no  l) no
19. Give an example of two 3-digit nonprime numbers that are relatively prime.

\[ 2^4 \times 11 = 176 \]
\[ 3^3 \times 13 = 351 \] for example

20. If possible, give a composite number that is relatively prime to 22.

It cannot have 2 or 11 as a factor so \(3 \times 13 = 39\) is relatively prime for example

21. a. Find \(128 + 494 + 381\) and check your answer by casting out nines.
   b. Compute \(23 \times 45\) and check your answer by casting out nines.

22. A result in number theory states that the product of any \(n\) consecutive positive integers is divisible by the product of the first \(n\) positive integers. For example, \(5 \times 6 \times 7 \times 8 = 1680\). The theorem asserts that 1680 is divisible by \(1 \times 2 \times 3 \times 4\) or 24. Verify that 1680 is divisible by 24, using divisibility rules.

Compute the product of another 4 consecutive positive integers and check to see if the product is divisible by 24.

Demonstrate this theorem for some example you choose for \(n = 5\).

\[ 7 \times 8 \times 9 \times 10 = 5040 \text{, so divisible by } 1 \times 2 \times 3 \times 4 = 24 \text{ } \Rightarrow \text{ check by using rules for } 3\text{ and } 8\]

3-ok sum of digits = 9 8-ok last 3 digits = 040 \(\div 8 = \)

23. A result in number theory states that if \(p\) is a prime number and \(n\) is a positive integer, then \(n^p - n\) is divisible by \(p\). Demonstrate this for 3 cases where you choose \(n\) and \(p\).

If \(p = 5\) and \(n = 4\), \(4^5 - 4\) should be divisible by 5. 
\(45 = 1024\) \(\div 5 = 1020\)

1020 is divisible by 5.

24. a. “I am a 3-digit number.
   I am not a multiple of 2.
   7 is not one of my factors, but 5 is.
   I am less than 125.
   Who am I?"'

b. Make up a “Who am I?” involving number theory vocabulary and ideas.

\[ 3 \div 3 \times 5 \div 11 \]

\[ 3, 5 = 15 \text{ no} \]
\[ 3, 5 = 45 \text{ no} \]
\[ 3, 5 = 15 \text{ no} \]
\[ 3, 5 = 75 \text{ no} \]
\[ 3, 5 = 55 \text{ no} \]
\[ 3, 11 = 165 \text{ no} \]
\[ 3, 13 = 65 \text{ no} \]
\[ 3, 17 = 85 \text{ no} \]
\[ 5, 19 = 95 \text{ no} \]
\[ 5, 23 = 115 \]
Learning Exercises for Section 11.4

1. List four factors of each of the following numbers:
   a. \(2^3\)  
b. \(27 \times 49\)  
c. 12  
d. 15  
e. 108  
f. 125  
g. 72

   a) 1, 2, 4, 8  
b) 1, 3, 9, 27, 7, 49  
c) 1, 2, 3, 4, 6, 12  
d) 1, 3, 5, 15  
e) 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108  
f) 1, 5, 25, 125  
g) 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72

2. List four nonzero multiples of each of the following. You may leave them in factored form.
   a. 8  
b. \(27 \times 49\)  
c. \(2^2 \times 3\)  
d. 15  
e. 108  
f. 125  
g. 72

   a) 8, 16, 24, 32  
b) \(27 \times 49\), \(27^2 \times 49\), \(27^2 \times 49^2\), \(27 \times 49^2\)  
c) \(2^2 \times 3\), \(2^3 \times 3\), \(2^3 \times 3^2\), \(2^3 \times 3^2\)  
d) 15, 30, 45, 60  
e) 108, 216, 324, 432  
f) 125, 250, 375, 500  
g) 72, 144, 216, 288

3. As you know, the Earth takes 365 "earth" days, that is, 1 year, to make one revolution around our sun. It takes Saturn about 30 earth years to make one revolution around the sun. Jupiter takes 12 earth years to do so, and Mars about 2 earth years. If Earth, Saturn, Jupiter, and Mars are aligned when they begin, in approximately how many years does it take all four planets to be aligned again?

\[
\text{LCM} \left( \frac{30}{2}, \frac{12}{2}, \frac{12}{2} \right) = 2^2 \times 3 \times 5 = 60
\]

4. You have probably noticed that when you buy hot dogs and hot dog buns, the packages do not contain the same quantities. Make up a story problem that gives quantities of hot dogs in a package and quantities of hot dog buns in a package and asks, "How many packages of each do you buy so that there are the same number of hot dogs as hot dog buns?" Would the solution involve LCM or GCF? Why?

Looking for LCM since you need the total amounts of both to match
5. For each of the following groups of numbers, find the least common (nonzero) multiple. You may leave your answers in factored form or write them out.
   a. 72 and 108  
   b. 144 and 150  
   c. 72 and 90  
   d. 72 and 144  
   e. 144 and 567  
   f. 90 and 567  
   g. 108 and 90  
   h. 150 and 350  
   i. 150, 35, and 270

6. For each of the following groups of numbers, find the greatest common factor. You may leave your answers in factored form or write them out.
   a. 72 and 108  
   b. 144 and 150  
   c. 72 and 90  
   d. 72 and 144  
   e. 144 and 567  
   f. 90 and 567  
   g. 108 and 90  
   h. 150 and 350  
   i. 150, 35, and 270

a) \[ \text{LCM}(72, 108) = 2^3 \cdot 3^3 = 216 \]  
   \[ \text{GCF}(72, 108) = 2^2 \cdot 3^2 = 36 \]  

b) \[ \text{LCM}(144, 150) = 2^4 \cdot 3^2 \cdot 5^2 = 3600 \]  
   \[ \text{GCF}(144, 150) = 2^2 \cdot 3 = 18 \]  

c) \[ \text{LCM}(72, 90) = 2^3 \cdot 3^2 \cdot 5 = 360 \]  
   \[ \text{GCF}(72, 90) = 2 \cdot 3^2 = 18 \]  

d) \[ \text{LCM}(72, 144) = 2^4 \cdot 3^2 = 144 \]  
   \[ \text{GCF}(72, 144) = 2^3 \cdot 3^2 = 72 \]  

e) \[ \text{LCM}(144, 567) = 2^4 \cdot 3^4 \cdot 7 = 9072 \]  
   \[ \text{GCF}(144, 567) = 3^2 = 9 \]  

f) \[ \text{LCM}(90, 567) = 2 \cdot 3^4 \cdot 5 \cdot 7 = 5670 \]  
   \[ \text{GCF}(90, 567) = 3^2 = 9 \]  

g) \[ \text{LCM}(108, 90) = 2^2 \cdot 3^3 \cdot 5 = 540 \]  
   \[ \text{GCF}(108, 90) = 2 \cdot 3^2 = 18 \]  

h) \[ \text{LCM}(150, 350) = 2 \cdot 3 \cdot 5^2 \cdot 7 = 1050 \]  
   \[ \text{GCF}(150, 350) = 2 \cdot 5^2 = 50 \]  

i) \[ \text{LCM}(150, 35, 270) = 2 \cdot 3^3 \cdot 5^2 \cdot 7 = 9450 \]  
   \[ \text{GCF}(150, 35, 270) = 5 \]
7. Give three (nonzero) multiples of each of the following (leave them in factored form).

a. \(2^3 \cdot 3\)  
b. \(3^2 \cdot 7^3\)  
c. \(3^2 \cdot 7^3 \cdot 11^5 \cdot 19^2\)

\(a) \ 2^3 \cdot 3, 2^4 \cdot 3, 2^3 \cdot 3^2, 2^4 \cdot 3^2\)  
\(b) \ 2^3 \cdot 7^3, 2^4 \cdot 7^3, 2^3 \cdot 7^4\)  
\(c) \ 3^2 \cdot 7^3, 11^5 \cdot 19^2, 3^3 \cdot 7^3 \cdot 11^5 \cdot 19^2\)

8. In each part, find the least common (nonzero) multiple of the numbers given. Then find the greatest common factor for each part.

a. \(m = 52 \cdot 7^3, n = 5 \cdot 13^2\)  
b. \(m = 37^4 \cdot 47^5, n = 37^6 \cdot 47^3 \cdot 71\)  
c. \(m = 7^2 \cdot 26, n = 2^2 \cdot 11 \cdot 17\)  
d. \(m = 10125, n = 26730\)  
e. \(m = 23 \cdot 7^2, n = 2 \cdot 5^3 \cdot 7, p = 32 \cdot 5 \cdot 7\)  
f. \(m = 6x^3y^5z^4, n = 10xy^6z^4\)

\(a) \ \text{LCM} = 5^2 \cdot 7^3 \cdot 13^2, \ \text{GCD} = 5\)  
\(b) \ \text{LCM} = 37^6 \cdot 47^5 \cdot 67^6 \cdot 71, \ \text{GCD} = 37^4 \cdot 47^5\)  
\(c) \ \text{LCM} = 2^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17, \ \text{GCD} = 2\)  
\(d) \ 10125 = 5^3 \cdot 74\)  
\(26730 = 2 \cdot 3^5 \cdot 5 \cdot 11\)  
\(\text{LCM} = 2 \cdot 3^5 \cdot 5^3 \cdot 74 \cdot 11, \ \text{GCD} = 5\)  
\(e) \ \text{LCM} = 2^3 \cdot 3^2 \cdot 5^3 \cdot 7^2, \ \text{GCD} = 7\)  
\(f) \ \text{LCM} = 30x^2y^6z^{12}, \ \text{GCD} = 2 \cdot xy^5z^4\)

9. \(0\) is a common multiple of 16 and 12. Why is the least common multiple defined as the smallest nonzero value?

\(0\) is the common multiple of all numbers, but not useful for fractions or other applications nor is it unique for any set of numbers.
10. Jogger A can run laps at the rate of 90 seconds per lap. Jogger B can run laps on the same track at the rate of 2 minutes per lap. If they start at the same place and time, and run in the same direction, how long (in time) will it be before they are at the starting place again, at the same time?

Jogger A is at start at
\[0, 90, 180, \ldots\] seconds
Jogger B is at start at
\[0, 120, 240, \ldots\] seconds
\[\text{LCM}(90, 120) = 360\]
so after 360 seconds (6 minutes)
both are at the start
Jogger A did 4 laps, Jogger B did 3 laps

11. You cut a yellow cake into 6 pieces, and a chocolate cake of the same size into 8 pieces. Then you find out that you were supposed to cut both cakes into the same (unspecified) number of pieces! As a number theory student, what do you do?

\[\text{LCM}(6, 8) = 24\]
so cut yellow cake's pieces into 4 parts
and chocolate cake's pieces into 3 parts

12. A machine has two meshing gears. One gear has 12 teeth and another gear has 30 teeth. After how many rotations are both gears back to their original position?

\[\text{LCM}(12, 30) = 60\]
Gear with 12 teeth will rotate 5 times
Gear with 30 teeth will rotate 2 times

13. Paper plates are sold in packages of 25. Paper bowls are sold in packages of 40. Plastic spoons are sold in packages of 20. How many packages of each do you need to buy to have the same number of plates, bowls, and spoons?

\[\text{LCM}(25, 40, 20) = 200\]
need 8 packages of plates, 5 packages of bowls
and 10 packages of spoons
14. A paint manufacturer produces base in 25-gallon drums and color in 10-gallon drums. A company wants to order stock for a dark blue paint mixture for which one gallon of color requires one gallon of base. How many of each should they order so that the mix comes out without any unused base or color left unmixed?

\[ \text{LCM} \ (25, 10) = 50 \]

2 drums of base and 5 drums of color

15. As in Exercise 14, a paint manufacturer produces base in 25-gallon drums and color in 10-gallon drums. A company wants to order stock for a dark blue paint mixture for which one gallon of color requires four gallons of base. How many of each should they order so that the mix comes out without any unused base or color left unmixed?

1 drum of color = 10 gallons
4 drums of base

\[ \text{lcm} \ (25, 40) = 200 \]

8 drums of base and 5 drums of color

16. The principal says that the sixth graders are raising money for a field trip by selling cups with the school logo. He said that last week they raised $414 and the week before they raised $543. You were going to ask what price they were selling for but realized you could figure out the price from what he had just told you. What was the likely selling price of the caps?

\[ \text{GCF} \ (414, 543) = 3 \text{ per cup} \]

414 = 2 * 3^2 * 23
543 = 3 * 181

17. The principal says the seventh graders raised money for a field trip to Washington, D.C., by washing cars for a fixed price the past three weekends. He said that last weekend they raised $198 and two weekends ago they raised $252. Three weekends ago they raised $385. What was the price of the car wash?

\[ \text{GCF} \ (198, 252, 385) = 31 \]

198 = 2 * 3^2 * 11
252 = 2^2 * 3^2 * 7
385 = 5 * 7 * 11

18. You want to explore the concept of scale factor with your students in an activity in which they will create a small-scale earth and sun to show their relative sizes. The sun is approximately 1,400,000 km in diameter and the earth is approximately 12,800 km in diameter. Because you want them to initially use only whole numbers, find the GCF to know what the scale factor should be.

\[ \text{GCF} \ (1400000, 12800) = 2^6 * 5^2 = 1600 \text{ km} \]

1,400,000 = 2 * 7 * 10^5 = 27(2^5, 5^5)
12,800 = 128 * 10^2
= 2^7(2^5, 5^2)
19. You want to explore the concept of scale factor with your students. You will work with your students to draw a scale drawing of your classroom. Your classroom has dimensions 216 inches by 282 inches. If you want them to use the smallest whole numbers possible, what should be the scale factor?

\[ \text{GCF}(216, 282) = 2 \cdot 3 = 6 \]

\[ 216 = 2^3 \cdot 3^3 \]

\[ 282 = 2 \cdot 3 \cdot 47 \]

20. Find several values of \( m \) and \( n \) such that \( 27^m = 9^n \).

\[ 27^m = 3^{3m} \quad 9^n = 3^{2n} \]

So \( 3m = 2n \)

1st \( m = 2, \quad n = 3 \)

2nd \( m = 4, \quad n = 6 \)

etc.

21. Write the simplest form for each fraction, using the greatest common factor:

a. \( \frac{135}{150} \)

b. \( \frac{36}{48} \)

c. \( \frac{84}{100} \)

d. \( \frac{180}{160} \)

e. \( \frac{2^3 \cdot 3^5}{2^4 \cdot 3^3} \)

\[ \text{a) GCF}(135, 150) = 15 \quad \text{b) GCF}(36, 48) = 12 \quad \text{c) GCF}(84, 100) = 2 \]

\[ \frac{9}{10} \quad \frac{3}{4} \quad \frac{21}{100} \]

d. \( \text{GCF}(180, 160) = 20 \)

e. \( \text{GCF}(2^3 \cdot 3^5 \cdot 2^3 \cdot 3^3) \)

\[ \frac{3^2}{2} \]

22. Use the phrase “relatively prime” to describe your final answers to parts a, b, c, and d in Exercise 21.

\[ \text{all are in simplest form since the numerator is relatively prime to the denominator in each case} \]
23. Use the least common multiple in calculating these sums and differences:

a. \[ \frac{7}{24} + \frac{11}{18} \]

b. \[ \frac{13}{8} - \frac{5}{4} \]

c. \[ \frac{14}{15} + \frac{7}{10} - \frac{1}{4} \]

d. \[ \frac{x^2}{y^2} + \frac{2x^2}{y} \]

e. \[ \frac{6.3 \times 10^4}{2 \times 10^2} + \frac{7.2 \times 10^5}{3 \times 10^3} \]

\[ \text{a)} \quad \frac{21}{72} + \frac{44}{72} = \frac{65}{72} \quad \text{b)} \quad \frac{13}{8} - \frac{10}{8} = \frac{3}{8} \]

\[ \text{c)} \quad \frac{56}{160} + \frac{42}{160} - \frac{15}{160} = \frac{83}{160} \]

\[ \text{d)} \quad \frac{x^3}{y^2} + \frac{2x^2y}{x^2y^2} = \frac{x^3 + 2x^2y}{x^2y^2} \]

\[ \text{e)} \quad \frac{18.9 \times 10^5}{6 \times 10^2} + \frac{14.4 \times 10^5}{6 \times 10^5} = \frac{33.3 \times 10^5}{6 \times 10^5} \]

24. For 12 and 16, the greatest common factor is 4 and the least common multiple is 48. Compare 12 \times 16 and 4 \times 48. Try other pairs of numbers to see whether their greatest common factor and least common multiple are related the same way.

Another pair is 24 and 360
GCF = 12 and LCM = 72
12 \times 72 = 864 \quad \text{and} \quad 24 \times 360 = 864

25. If your calculator allows you to enter and simplify fractions, use it to find simpler, and eventually the simplest, fractions for the following. (If you do not have a calculator, try them with paper and pencil.)

\[ \text{a)} \quad \frac{1280}{1440} = \frac{8}{9} \quad \text{b)} \quad \frac{8530}{47250} = \frac{853}{4725} \quad \text{c)} \quad \frac{2730}{10000} = \frac{17}{425} \]

26. Notice that the number 3 is a common factor of 84 and 72 and 3 is also a factor of 84 - 72 = 12.

a. Conjecture: If \( x \) is a common factor of \( m \) and \( n \), then \( x \) is also a factor of \( m - n \).

b. Test the conjecture by considering at least four examples of \( m \) and \( n \).

\[ \text{If} \quad m = 48 \quad \text{and} \quad n = 36, \quad \text{is a factor of both} \quad \text{and} \quad 6 \quad \text{is a factor of} \quad 48 - 36 = 12 \]

c. Find simpler names for \( \frac{147}{150} \) and \( \frac{111}{126} \), with help from part (a).

d. Determine whether this conjecture is relevant to \( \frac{81}{89} \), or \( \frac{48}{55} \), or \( \frac{112}{127} \). Explain. (Note: they cannot be reduced.)

\[ 147 = 49 \quad \text{so} \quad 3 \quad \text{is a common factor} \quad \text{so} \quad 6 \quad \text{is a common factor} \quad \frac{126 - 111}{126} = \frac{15}{42} \]
27. Practice: Write the following fractions in simplest form.

a. \( \frac{15}{35} \)  
   \( \frac{3}{7} \)  

b. \( \frac{28}{54} \)  
   \( \frac{14}{27} \)  

c. \( \frac{150}{350} \)  
   \( \frac{3}{7} \)  

d. \( \frac{12}{144} \)  
   \( \frac{1}{12} \)  

e. \( \frac{150}{567} \)  
   \( \frac{50}{189} \)  

f. \( \frac{15}{40} \)  
   \( \frac{3}{8} \)  

g. \( \frac{64}{512} \)  
   \( \frac{1}{8} \)  

h. \( \frac{21}{49} \)  
   \( \frac{3}{7} \)  

i. \( \frac{2223}{4536} \)  
   \( \frac{247}{504} \)  

28. Practice: Use the least common multiple of the denominators to calculate these sums and differences. For extra practice, also write the answer in simplified form.

\[ a. \frac{39}{144} + \frac{35}{108} = \frac{30}{24} + \frac{25}{27} = \frac{3 \times 30}{24} + \frac{2 \times 35}{27} = \frac{90}{72} + \frac{70}{81} = \frac{2223}{4536} \]

\[ b. \frac{25}{72} + \frac{81}{567} = \frac{25 \times 7}{21} + \frac{81 \times 7}{7 \times 81} = \frac{175 + 567}{147} = \frac{742}{147} \]

\[ c. \frac{36}{108} + \frac{110}{169} = \frac{36 \times 169}{18} + \frac{110 \times 108}{169} = \frac{6072 + 11880}{2904} = \frac{18952}{2904} = \frac{195}{26} \]

\[ d. \frac{15}{39} + \frac{110}{169} = \frac{15 \times 169}{3 \times 13} + \frac{110 \times 39}{13} = \frac{2535 + 410}{3 \times 13} = \frac{2945}{3 \times 13} = \frac{525}{507} \]

\[ e. \frac{169}{500} + \frac{169}{650} = \frac{169 \times 650 + 169 \times 500}{2 \times 5 \times 13} = \frac{109350 + 84500}{2 \times 5 \times 13} = \frac{193850}{2 \times 5 \times 13} = \frac{38770}{500} \]

\[ f. \frac{126}{504} + \frac{98}{770} = \frac{126 \times 770 + 98 \times 504}{2 \times 3 \times 7 \times 11} = \frac{95220 + 49248}{2 \times 3 \times 7 \times 11} = \frac{144468}{2 \times 3 \times 7 \times 11} = \frac{10458}{27720} \]

29. As a third-grade teacher, you are designing a measurement lesson. You decide to start with a measuring lesson using nonstandard units. You create a measuring tool (a new “ruler”) that will allow your students to measure classroom items and have the measure in whole units. The desks in your classroom are 27 inches wide and 36 inches long. What is the largest unit you can create that will allow this desk to be measured in whole units?

\[ \text{GCF}(27, 36) = 9 \text{ inches so rules should be } 9" \]