LAB 7: MODELING WITH EXPONENTIAL and LOG FUNCTIONS

Objectives:

1. Use natural logarithms to develop exponential models.
2. Make predictions using exponential models.
3. Compare growth rates of exponential models.

Reference Topics: Properties of Logarithms
Solving Exponential and Logarithmic Equations
Exponential and Logarithmic Models

Discussion:

As you have seen, data sets often fall into patterns that are nonlinear. However, it is not always easy to tell just what the nonlinear pattern is. You can determine if data follows an exponential pattern by looking for linear relationships between independent variables and the logarithms of dependent variables.

Consider the graph shown below.

The data does not seem to follow a linear pattern. But just what nonlinear pattern does the data best follow? Quadratic? Cubic? Exponential?

You can determine if the data falls into an exponential pattern by studying the semi-log graph of the data. This graph has the input values along the x-axis and the natural logarithm of the output values along the y-axis. That is, it is a graph of the (x, ln y) pairs. The semi-log graph of the data is shown below.
Notice that the semi-log data has a linear pattern. This means that you can write the following equation:

\[
\ln y = mx + b
\]

Solving for \(y\) you get the following:

\[
\begin{align*}
\ln y &= mx + b \\
\exp^{\ln y} &= \exp^{mx + b} & \text{Exponentiate each side of the equation.} \\
y &= \exp^{mx + b} \\
y &= \exp^{mx} \exp^{b} & \text{Property of exponents.} \\
y &= \exp^{b} \exp^{mx} & \text{Commutative property for multiplication.} \\
y &= Ce^{mx} & \text{Substitute } C \text{ for } \exp^{b} \text{ since } \exp^{b} \text{ is just a constant.}
\end{align*}
\]

Thus, the original data - the (x, y) pairs - must follow an exponential pattern.

In general, if the semi-log (x, \(\ln y\)) data follows a linear pattern, then the original (x, y) data follows an exponential pattern. If the semi-log (x, \(\ln y\)) data does not follow a linear pattern, then the original (x, y) data does not follow an exponential pattern.

**Activity 1**

Consider the following data for United States population. The left-hand table gives the colonial population from 1700 to 1780 and the right-hand table gives the population after the United States became an independent nation, from 1790 to 1870.

(Source: Department of Commerce, Bureau of the Census)

<table>
<thead>
<tr>
<th>Year (t)</th>
<th>U.S. Population (P) in millions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1700</td>
<td>0.251</td>
</tr>
<tr>
<td>1710</td>
<td>0.332</td>
</tr>
<tr>
<td>1720</td>
<td>0.466</td>
</tr>
<tr>
<td>1730</td>
<td>0.625</td>
</tr>
<tr>
<td>1740</td>
<td>0.907</td>
</tr>
<tr>
<td>1750</td>
<td>1.171</td>
</tr>
<tr>
<td>1760</td>
<td>1.594</td>
</tr>
<tr>
<td>1770</td>
<td>2.148</td>
</tr>
<tr>
<td>1780</td>
<td>2.780</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Year (t)</th>
<th>U.S. Population (P) in millions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1790</td>
<td>3.929</td>
</tr>
<tr>
<td>1800</td>
<td>5.309</td>
</tr>
<tr>
<td>1810</td>
<td>7.240</td>
</tr>
<tr>
<td>1820</td>
<td>9.639</td>
</tr>
<tr>
<td>1830</td>
<td>12.866</td>
</tr>
<tr>
<td>1840</td>
<td>17.069</td>
</tr>
<tr>
<td>1850</td>
<td>23.192</td>
</tr>
<tr>
<td>1860</td>
<td>31.443</td>
</tr>
<tr>
<td>1870</td>
<td>39.818</td>
</tr>
</tbody>
</table>

a. Plot all the data for 1700 - 1870. Use t = 0 to represent the year 1700, t = 10 to represent 1710, etc. You will need to adjust your scale.

What does the independent variable, t, represent?

What does the dependent variable, P, represent?
b. Let \( y = \ln P \) (the natural log of the population). Plot the \((t, \ln P)\) pairs for 1700 - 1870. Do this by taking the natural log of each \( P \) value in part a. Adjust your scale so that you have a good view of the semi-log data. What is the general shape of the graph?

c. What does your answer to part \( b \) imply about the pattern of the original population data? Explain.

d. Write an equation of the form \( y = mt + b \) that fits the semi-log graph in part \( b \).

\[
y = \]

Graph the equation to see how well it fits the semi-log data you graphed in part \( b \). Refine your model if necessary. You should be comfortable that you have a “best fit” model before you continue.

e. Since your model in part \( d \), is a model for the \((t, \ln P)\) pairs, \( y = \ln P \).

Thus, \( \ln P = \)

f. Solve the equation in part \( e \) for \( P \) by exponentiating both sides of the equation to create a model of the original data. Write the model in the form \( P = e^{mt + b} \)

\( P = \)

g. Rewrite as an exponential model of the form \( P = Ce^{mt} \) as demonstrated in the discussion at the beginning of this lab.

\( P = \)

h. Graph the equation with the original data to see how well it fits. You will need to adjust your scale.

i. What is the domain for this model?

j. Use this model to estimate the population of the United States in 1776

1900

2000

k. Is the growth rate of the population constant? Explain.
1. Use your population model to estimate the growth rate of the population at year 1750 \((t = 50)\) by finding the average rate of change between 1750 and 1751.

Use your population model to estimate the growth rate of the population at year 1800 \((t = 100)\) by finding the average rate of change between 1800 and 1801.

Is the population growing at the same rate in 1750 as it is in 1800?

m. The **relative growth rate at time** \(t\) **is given as a percent of the population living at time** \(t\) **and is defined as**

\[
\frac{\text{growth rate at time } t}{\text{population at time } t} \times 100\%
\]

Estimate the **relative** growth rate at year 1750. Give your answer to the nearest 0.1%.

Estimate the **relative** growth rate at year 1800. Give your answer to the nearest 0.1%.

What do you notice about the **relative** growth rates for 1750 and 1800?

n. Use the results in part m to help you estimate the relative growth rate of the population during 1865.

o. Convert the % relative growth rate to a decimal number.

The above results are estimates of the relative growth rate for your function. You can find the relative growth rate more precisely without all the above computation by looking at your exponential model for the population. What part of your model is approximately the same as your estimate of relative growth rate in decimal form? Explain.

p. According to your model, what is the **relative** growth rate (in %) for the U.S. population between 1700 and 1870?
Activity 2

a. Test your model with the following data for 1880 - 1990. Plot the new data, still with \( t = 0 \) corresponding to 1700. Do you think the model you developed in Activity 1 fits this data well? Explain.

<table>
<thead>
<tr>
<th>Year (t)</th>
<th>U.S. Population (P) in millions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1880</td>
<td>50.155</td>
</tr>
<tr>
<td>1890</td>
<td>62.948</td>
</tr>
<tr>
<td>1900</td>
<td>75.995</td>
</tr>
<tr>
<td>1910</td>
<td>91.972</td>
</tr>
<tr>
<td>1920</td>
<td>105.711</td>
</tr>
<tr>
<td>1930</td>
<td>122.775</td>
</tr>
<tr>
<td>1940</td>
<td>131.669</td>
</tr>
<tr>
<td>1950</td>
<td>150.697</td>
</tr>
<tr>
<td>1960</td>
<td>179.323</td>
</tr>
<tr>
<td>1970</td>
<td>203.302</td>
</tr>
<tr>
<td>1980</td>
<td>226.546</td>
</tr>
<tr>
<td>1990</td>
<td>248.710</td>
</tr>
</tbody>
</table>

(Source: Dept. of Commerce, Bureau of the Census)

b. Plot all the semi-log data ((t, \( \ln P \)) pairs) for 1700 - 1990 along with the semi-log model from Activity 1 part d. Turn in a labeled graph. Looking at the graph, does the model fit the semi-log data for 1880-1990? Explain.

c. Create a new linear model that fits the semi-log data for 1880 - 1990. 

\[
\ln P =
\]

NOTE: You should be comfortable that you have a “best fit” model before you continue.
d. Use the result from part c to create an exponential model that fits the data for 1880 - 1900. Graph the model to see how well it fits the data. Write your model in the form $P = Ce^{mt}$.

$$P =$$

e. What is the domain of this model?

f. Use the model to estimate the population in the United States in

1900

1990

2000

2010

g. In what year does your model predict the U.S. population will reach

300 million?

400 million?

h. According to your model, what was the relative growth rate (in %) for the U.S. population between 1880 and 1990?

Is this rate the same as that for the population data for the years 1700 to 1870?

Do you expect the population to continue to grow at this rate in the future? Justify your answer.

i. Compare the results from the two models you have created. Which model is better for predicting the U.S. population in the year 2000? Why?
Activity 3

Notice that the growth rate of the U.S. population appears to be decreasing. You can observe this from the \((t, \ln P)\) pairs that you graphed. When you look at all these pairs together, you should see that the points do not fall on a straight line. (Check this.) This indicates that the population of the U.S. is not following a pattern of unlimited exponential growth (Malthusian growth). When some physical constraint tends to limit the ultimate growth of the population a more realistic model for population growth is a logistics model. A logistics model or sigmoidal curve increases exponentially at first, but then levels off with the passage of time. (See figure below.)

3. Use a logistics model to estimate the U. S. population.
   a. What factors do you think would limit population growth in the United States? (List several.)
   
   b. Graph the following logistics model for U. S. population along with the data for 1700 - 1990.

   \[
   P = \frac{400}{1 + 475e^{-0.023t}} \quad \text{with } t = 0 \text{ representing 1700 and } P \text{ in millions}
   \]

   c. What does the independent variable, \(t\), represent?

   d. What does the dependent variable, \(P\), represent?

   e. According to this model, what is the maximum U. S. population possible?
f. Use this model to estimate the U. S. population in
   1900
   1990
   2000
   2010

g. According to this model, what is the first year in which the U.S. population will exceed 300 million?

Activity 4

a. Turn in a labeled graph showing the population data for 1700-1990 along with the three population models you have used in this lab (from parts 1g, 2d, and 3b).

b. Write a report (on separate paper) in which you do the following:
   Compare the three population models you have used in this lab.
   Discuss strengths and weaknesses of each model.
   Discuss and compare the growth rates and relative growth rates of the models.
   Predict the U.S. population in 2000 and 2010 based on your work with the models in this lab, and explain how and why you made those predictions.