LAB 10: FINDING AREAS USING PARTIAL SUMS

Objectives:
1. Find the sum of a finite number of terms in a sequence.
2. Find areas using partial sums.

Reference Topic: Sequences and Partial Sums

Discussion:
You know how to find the area of many geometric figures using algebraic formulas. For example, the area of a square with side of length $s$ is $A = s^2$.

\[ s \quad A = s^2 \quad s \]

The area of a rectangle with length $l$ and width $w$ is $A = lw$.

\[ w \quad A = lw \quad l \]

The area of a trapezoid with parallel sides of lengths $b_1$ and $b_2$ and perpendicular side of length $h$ is $A = \frac{1}{2} h(b_1 + b_2)$.

\[ b_1 \quad b_2 \quad h \]

However, there are no algebraic formulas for the area of a region with one curved side such as the one below.

In this lab you will estimate areas of regions with curved boundaries by summing areas of trapezoids that approximate the regions.
Finding Partial Sums

You can use DERIVE or the TI-82 to find the sum of the first n terms in any sequence.

1. Find the sum of the first 50 terms in the following sequence:

\[ 1, \, 4, \, 9, \, 16, \, 25, \ldots, \, n^2, \]

To compute the partial sum of terms in a sequence, use the \texttt{sum seq} statement. The format of the \texttt{sum seq} statement is

\[
\texttt{sum seq(expression, variable, begin, end, 1)}
\]

and corresponds to

\[
\sum_{\text{variable} = \text{begin}}^{\text{end}} \text{expression}
\]

Make sure you are in the Home window.

\textbf{TI-82} > Press \texttt{2nd STAT} for \texttt{LIST}  \\
> Press \texttt{5} for \texttt{sum}  \\
- You should see \texttt{sum} in the Home window.

\textbf{TI-82} > Press \texttt{2nd STAT} for \texttt{LIST}  \\
> Press \texttt{5} for \texttt{seq(}  \\
- You should see \texttt{sum seq(} in the Home window.

\textbf{TI-82} > Enter: \texttt{X‰,X,1,50,1)} for the sequence \(x^2\), \textbf{with } \textit{x} \textbf{going from } \textit{1} \textbf{to } \textit{50} \textbf{at } \textit{1} \textbf{unit intervals}  \\
- You should see \texttt{sum seq(X‰,X,1,50,1)} \textit{50}  \\
\textit{which means the same thing as } \sum_{x = 1}^{50} x^2

> To find the sum, press \texttt{ENTER}.

\[ 1 + 4 + 9 + 16 + 25 + \ldots + 49^2 + 50^2 = \text{______________} \]
To compute the partial sum of terms in a sequence, use the Calculus Sum commands.

> Type: C for Calculus

> Type: S for Sum
- You should see the prompt:
  \[ \text{CALCULUS SUM expression:} \]

> Type: \( i^2 \)
- You should see the prompt:
  \[ \text{CALCULUS SUM variable:} \ i \]

**DERIVE**

> Press ENTER
- You should see the prompt:
  \[ \text{CALCULUS SUM:} \ \text{Lower limit:} \ 1 \ \text{Upper limit:} \ n \]

> Press the TAB key and change the upper limit to 50.

> Press ENTER.

What do you see on the screen?

To find this sum, use the approx or simplify or command.

\[ 1 + 4 + 9 + 16 + 25 + \ldots + 49^2 + 50^2 = \_\_\_\_\_\_\_\_ \]

2. Use the Calculus Sum commands on DERIVE or \texttt{sum seq(} statement on the TI-82 to find the following partial sums.

\[ \begin{align*}
\text{a.} \quad & 2k+3 \\
& \text{for } k = 1 \\
\text{b.} \quad & 100(1.05)^i \\
& \text{for } i = 1 \\
\text{c.} \quad & \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
& \text{First write this in summation notation in the space below.}
\end{align*} \]

\[ \text{d.} \quad \text{f(x \ 1)} \text{ where } f(x) = 2x^2 \ 3x + 4 \]
\[ \text{x = 1} \]

HINT: First define the function so that you can use function notation in the summation.
Application of Partial Sums: Finding Areas

To find the area of a trapezoid, you can use the formula $A = \frac{1}{2} h(b_1 + b_2)$ where $h$, $b_1$, and $b_2$ are shown in the figure below.

1. You can find the area of irregular regions that can be divided into several trapezoids by finding the area of each trapezoid and adding their areas. Find the area of the following figures.

2. You can also use trapezoids to estimate the area of a figure with a curved side. For example, consider the area shown, bounded by the x-axis, the y-axis, the line $x=4$ and the parabola $f(x) = 25 - x^2$. 
You can estimate the area by dividing the approximate area into trapezoids as in the figure shown to the right.

a. Make a table of points on the graph of f(x) for x = 0, 1, 2, 3 and 4. Write the resulting table in the space below.

b. Plot the points on DERIVE or the TI-82 using the connected mode.

**DERIVE**

Press the **Options** command in the Graphing window, then press **S** for **State**. Tab over to the **Mode** field and press **C** for **Connected**. Press **ENTER**, then **Plot**.

**OR**

Enter the x values in L1 and the f(x) values in L2. Then set up your **STATPLOT Plot1** window as follows:

**TI-82**

![Graph setup](image)

c. Make a copy of the graph of the line segments. Use the line segments to help you sketch 4 trapezoids that, taken together, approximate the original region. Turn in the graph with this lab.

d. Graph f(x) over the interval [0, 4]. Do the line segments seem to be a good approximation of the graph of f(x)? Explain.

e. Find the areas of the four trapezoids in part c and use them to estimate the area of the given region. Show your work below.
Note that to estimate the area of the region, you take the sum of the areas of a sequence of trapezoids.

\[ A = \sum_{i=1}^{n} A_i \]

where \( n \) is the number of trapezoids and \( A_i \) is the area of the trapezoid in the \( i \)th position from the left

Thus, you should be able to write this problem using summation notation and let the computer or calculator do all the computation for you. To do this, you need to find a formula for \( A_i \).

Note that \( A_i = \frac{1}{2} h(b_{i-1} + b_i) \), where \( b_{i-1} \) is the length of the left side of the \( i \)th trapezoid, \( b_i \) is the length of the right side of the \( i \)th trapezoid, and the height, \( h \), is the distance between \( b_{i-1} \) and \( b_i \).

In the example you have been working, the height of each trapezoid, \( h = \) ______

Note that the bases of the trapezoids are the function values at \( x = 0, 1, 2, 3 \) and 4. For example the area of the first trapezoid is

\[ A_1 = \frac{1}{2} h(b_0 + b_1) = \frac{1}{2} (1)[(0) + (1)] \]

Similarly, \( A_2 = \frac{1}{2} h(b_1 + b_2) = \) ________________

\[ A_3 = \] ________________

\[ A_4 = \] ________________

and thus \( A_i = \frac{1}{2} h(b_{i-1} + b_i) = \) ________________

Complete the following:

The area of the region, \( A = \sum_{i=1}^{4} A_i = \) ________________

Approximate the area of the region by finding the sum in part g using the \textbf{Calculate} \textbf{Sum} commands on the computer or the \texttt{sum} \texttt{seq}( statement on the calculator.
3. You can get a better estimate of the area by using more trapezoids. By using the Calculus Sum command on the computer or the \texttt{sum seq} statement on the calculator it is just as easy to compute the area of a large number of trapezoids as a small number. Repeat problem 2 above, but this time use 8 trapezoids. The steps below will help you set up the problem.

a. Make a table of points on the graph of \( f(x) \) corresponding to \( x = 0, .5, 1, 1.5, 2, 2.5, 3, 3.5 \) and 4. Write the resulting table in the space below.

b. Plot the points on DERIVE or the TI-82 using the connected mode. Be sure to choose an appropriate scale.

c. Make a copy of the graph of the line segments. Use the line segments to help you sketch 8 trapezoids that, taken together, approximate the original region. Turn in the graph with this lab.

d. Graph \( f(x) \) over the interval \([0, 4]\). Do the line segments seem to be a good approximation of the graph of \( f(x) \)? Explain.

e. When you divide the region into 8 trapezoids as in part c, what is the height, \( h \), of each trapezoid?

\[ h = \underline{\quad} \]

f. You should be able to write this problem using summation notation and let the computer do all the computation for you.

\[
A = \frac{1}{2} \sum_{i=1}^{n} h(b_{i-1} + b_i),
\]

where \( h \) is the height of each trapezoid and \( n \) is the number of trapezoids.

With 8 trapezoids, \( n = \underline{\quad} \)

The height of each trapezoid, \( h = \underline{\quad} \)
Note that the bases of the trapezoids are the function values at 
\( x = 0, .5, 1, 1.5, 2, 2.5, 3, 3.5 \) and 4. For example, the area of the 
first trapezoid \( (i = 1) \) is

\[
A_1 = \frac{1}{2} h(b_0 + b_1) = \frac{1}{2} (0.5) \left[ (0) + (0.5) \right]
\]

g. Complete the following table, giving the formula for the area of each 
of the 8 trapezoids in this problem. The last line asks you to look 
for a pattern and generalize by finding a formula relating the area of 
a trapezoid, \( A_i \), to its index or position number, \( i \).

<table>
<thead>
<tr>
<th>i</th>
<th>( A_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[ A_1 = \frac{1}{2} h(b_0 + b_1) = \frac{1}{2} (0.5) \left[ (0) + (0.5) \right] ]</td>
</tr>
<tr>
<td>2</td>
<td>( A_2 = )</td>
</tr>
<tr>
<td>3</td>
<td>( A_3 = )</td>
</tr>
<tr>
<td>4</td>
<td>( A_4 = )</td>
</tr>
<tr>
<td>5</td>
<td>[ A_5 = \frac{1}{2} h(b_4 + b_5) = \frac{1}{2} (0.5) \left[ (0.5(4)) + (0.5(5)) \right] ]</td>
</tr>
<tr>
<td>6</td>
<td>( A_6 = )</td>
</tr>
<tr>
<td>7</td>
<td>( A_7 = )</td>
</tr>
<tr>
<td>8</td>
<td>( A_8 = )</td>
</tr>
<tr>
<td>i</td>
<td>( A_i = )</td>
</tr>
</tbody>
</table>

g. Use the last line of the chart to help you complete the following:

\[
\sum_{i=1}^{n} A_i = \text{__________________________}
\]

h. Use DERIVE or the TI-82 to approximate the area of the region by 
finding the sum in part g.
Write an explanation to another lab team describing how to use trapezoids and summation to estimate the area of a region with a curved boundary. Illustrate your explanation by estimating the area of the shaded region shown below using

a. 3 trapezoids
b. 6 trapezoids
c. 30 trapezoids

Show all work including graphs and summation statements. Tell which part, a, b or c gives the best estimate.

The curved boundary is \( g(x) = 2x^4 - 11.4x^3 + 18.7x^2 - 6.1x + 2 \).