The Party Problem
An Introduction to Combinatorics

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Combinatorics is a field of math primarily concerned with four main topics of finite structures

- Enumeration - counting certain structures
- Existence of structures with certain properties
- Construction of these structures
- Optimization - finding a minimum of maximum parameter for these structures
The Party Problem

We would like to know how many people must be invited to a party where we can guarantee that there is a group of 3 people who either all are friends or all are strangers. We want to invite the minimum number of people because we would like to spend as little money as possible.
To clarify the problem we will make some assumptions:

- Every pair of people at the party is a pair of friends or strangers (not both)
- The stranger and friend relationships are symmetrical (nobody has amnesia)
We can model the relationships at our party with graph theory. A graph consists of a set of vertices and a set of edges between those vertices:
To solve our problem, we will represent each person at our party as a vertex on our graph. We will place a red edge between every pair of friends and a blue edge between every pair of strangers:
The Party Problem

This graph shows us that inviting 4 people is not enough to guarantee a group of 3 mutual friends or a group of 3 mutual strangers. What about 5 people?
The Party Problem

Five is also not enough people:
But 6 is enough people, and we can prove it. Isolate one person.
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No matter how we color the edges there are always at least 3 edges that are the same color.

![Graph diagram]
Suppose these three edges are red. Then if any of the edges between those three people are also red, we have the group of 3 friends that we wanted.
The only other case is that all three of these edges are blue, which gives us a group of three strangers:
Combinatorialists study the generalization of this problem in a field known as Ramsey Theory.

Let the Ramsey number $R(m, n)$ be the minimum number $k$ such that when the edges of the complete graph on $k$ vertices are colored with red and blue, there will always either be a red clique of size $m$ or a blue clique of size $n$.

To put this in the language of our problem we want to invite $k$ people so that we have a group of $m$ friends or a group of $n$ strangers.
Some known Ramsey numbers:

\[ R(3, 3) = 6 \]
\[ R(3, 4) = 9 \]
\[ R(4, 4) = 18 \]
\[ R(4, 5) = 25 \]
\[ 43 \leq R(5, 5) \leq 48 \]
\[ 102 \leq R(6, 6) \leq 165 \]
Notice that we don’t know the exact value for those last few. To check that $R(5, 5)$ is actually larger than 43, we would have to find a coloring of the edges of the graph that does not have a red or blue clique. The complete graph on 43 vertices has 903 edges. This means there are $2^{903}$ possible colorings. Even if we could check a trillion colorings per second, this would still take over $10^{242}$ years.
“Suppose aliens invade the earth and threaten to obliterate it in a year’s time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world’s best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.”

— Paul Erdős
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Thank you!