# The Party Problem

An Introduction to Combinatorics

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# Combinatorics

Combinatorics is a field of math primarily concerned with four main topics of finite structures

- Enumeration - counting certain structures
- Existence of structures with certain properties
- Construction of these structures
- Optimization - finding a minimum of maximum parameter for these structures

# The Party Problem

We would like to know how many people must be invited to a party where we can guarantee that there is a group of 3 people who either all are friends or all are strangers. We want to invite the minimum number of people because we would like to spend as little money as possible.

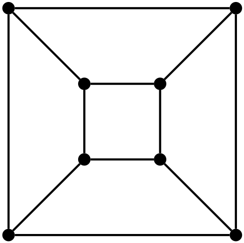
# The Party Problem

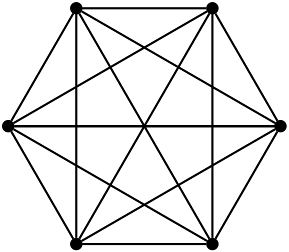
To clarify the problem we will make some assumptions:

- Every pair of people at the party is a pair of friends or strangers (not both)
- The stranger and friend relationships are symmetrical (nobody has amnesia)

# Graphs

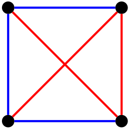
We can model the relationships at our party with graph theory. A **graph** consists of a set of vertices and a set of edges between those vertices:

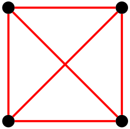




# The Party Problem

To solve our problem, we will represent each person at our party as a vertex on our graph . We will place a red edge between every pair of friends and a blue edge between every pair of strangers:





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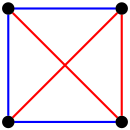
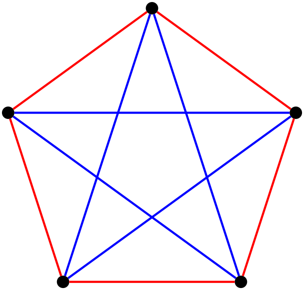


This graph shows us that inviting 4 people is not enough to guarantee a group of 3 mutual friends or a group of 3 mutual strangers. What about 5 people?

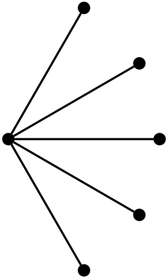
# The Party Problem

Five is also not enough people:



# The Party Problem

But 6 is enough people, and we can prove it. Isolate one person.



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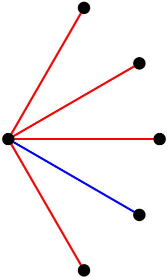
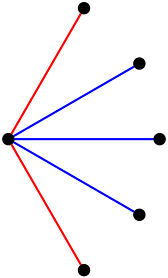


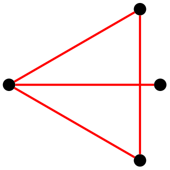
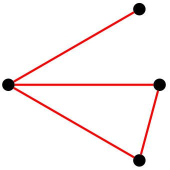
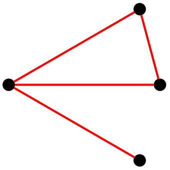
# The Party Problem

No matter how we color the edges there are always at least 3 edges that are the same color.



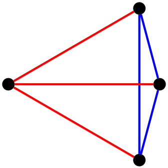
# The Party Problem

Suppose these three edges are red. Then if any of the edges between those three people are also red, we have the group of 3 friends that we wanted.

# The Party Problem

The only other case is that all three of these edges are blue, which gives us a group of three strangers:



Combinatorialists study the generalization of this problem in a field known as Ramsey Theory.

Let the Ramsey number $R(m, n)$ be the minimum number $k$ such that when the edges of the complete graph on $k$ vertices are colored with red and blue, there will always either be a red clique of size $m$ or a blue clique of size $n$.

To put this in the language of our problem we want to invite $k$ people so that we have a group of $m$ friends or a group of $n$ strangers.

# Ramsey Theory

Some known Ramsey numbers:

$R(3,3) = 6$

$R(3,4) = 9$

$R(4,4) = 18$

$R(4,5) = 25$

$43 \leq R(5,5) \leq 48$

$102 \leq R(6,6) \leq 165$

Notice that we don't know the exact value for those last few. To check that $R(5,5)$ is actually larger than 43, we would have to find a coloring of the edges of the graph that does not have a red or blue clique. The complete graph on 43 vertices has 903 edges. This means there are $2^{903}$ possible colorings. Even if we could check a trillion colorings per second, this would still take over $10^{242}$ years.

# Ramsey Theory

"Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack."

— Paul Erdős

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Thank you!