Determinants

Goal: To calculate determinants.

The determinant of a 2×2 matrix is...

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

take determinant

ex $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$

$$|A| = \det A = 1(-4) - (2)(3)$$
$$= -4 - 6$$
$$= -10$$

ex Find det A when $A = \begin{bmatrix} -2 & -3 \\ 5 & 4 \end{bmatrix}$

$$|A| = -8 + (+15)$$
$$= 7$$

ex Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & 1 & -4 \\ 2 & 3 & 6 \end{bmatrix}$. Find $|A|$.

$a_{12} = -3$

$a_{23} = -4$

$$|A| = 1 \cdot \begin{vmatrix} 1 & -4 \\ 3 & 6 \end{vmatrix} - (-3) \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} + 2 \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 1 \cdot (6 + (+12)) + 3\,(18 + (+8)) + 2\,(9 - 2)$$

$$= 18 + 3(26) + 2(7)$$

$$= 18 + 78 + 14$$

$$= 110$$

Definition of a 3x3 Determinant

To each entry in a 3×3 matrix, there corresponds a 2×2 sub-matrix. For example, the sub matrix associated with $\boxed{a_{12}}$ above is $\begin{bmatrix} 3 & -4 \\ 2 & 6 \end{bmatrix}$

$\overset{\text{row}}{} \; \overset{\text{column}}{}$
$a_{1,2}$ = entry in the 1st row, 2nd column A.

The determinant of this sub-matrix is called M_{12} (M stands for minor) called M_{12} (M stands for <u>minor</u>)

from last ex

$M_{12} = 26$

$M_{13} = 7$

$\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 9 - 2 = 7$

In terms of expanding about the first row, the determinant of matrix A is ...

$$|A| = a_{11}M_{11} \overset{\text{def}}{-} \overset{0}{a_{12}}M_{12} + a_{13}M_{13}$$

Notes:

(1) The <u>cofactor</u> for the ith, jth entry is a signed minor

entry in ith row jth column

$$C_{ij} = (-1)^{i+j} M_{ij}$$

last ex

$$\underset{i\ j}{C_{13}} = (-1)^{1+3}(7) = (-1)^4 7 = 1 \cdot 7 = 7$$

(2) The above method of finding a determinant is called "expansion

by cofactors about the first row."

③ You can expand about any row or column to get $|A|$, as long as you use the sign array...

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

ex Find $|A|$ from 1st ex by expanding about the 2nd column.

$$|A| = \begin{vmatrix} 1 & -3 & 2 \\ 3 & 1 & -4 \\ 2 & 3 & 6 \end{vmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$|A| = -(-3)\begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 3 & -4 \end{vmatrix}$$

$$= 3(18+8) + 1(6-4) - 3(-4-6)$$

$$= 3(26) + 2 - 3(-10)$$

$$= 78 + 2 + 30$$

$$= 110$$

$= 110$

Notes: Let A be a square matrix.

1. Find $|A|$ by expanding about the row or column with the most zero entries.

2. If a row or column of matrix A contains all 0's, then $|A| = 0$

3. $|AB| = |A|\,|B|$

4. The determinant of a triangular matrix is the product of the entries down the main diagonal.

$A = \begin{bmatrix} -2 & 0 \\ 3 & 8 \end{bmatrix}$

$|A| = -2(8) = -16$

$B = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 6 & 3 \\ 0 & 0 & 4 \end{bmatrix}$

$|B| = 0 - 0 + 4\begin{vmatrix} 3 & 1 \\ 0 & 6 \end{vmatrix}$

$= 4(18)$

$= 72$

$|B| = 3(6)(4) = 72$

ex calculate:

a) $\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = 2$

b) $\begin{vmatrix} 3 & -2 & 1 \\ 0 & 5 & 6 \\ 0 & 0 & 2 \end{vmatrix} = 30$

⑤ $|A^{-1}| = \dfrac{1}{|A|}$

Proof: $|A \cdot A^{-1}| \overset{③}{=} |A||A^{-1}|$

$|I| =$ "

$\dfrac{1}{|A|} = \dfrac{\cancel{|A|}|A^{-1}|}{\cancel{|A|}}$

$\dfrac{1}{|A|} = |A^{-1}|$ Done

⑥ A^{-1} exists iff $|A| \neq 0$

⑦ Multiplying a row or column of A by a constant, k, makes the determinant of the new matrix k times as big.

ex $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $\xrightarrow{\times 3}$ $B = \begin{bmatrix} 2 & 9 \\ 4 & 15 \end{bmatrix}$

$|A| = 10 - 12$
$= -2$

$|B| = 30 - 36$
$= -6 = 3|A|$