Section 11.3: Geometric Sequences and Series

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1. To find the nth term of a geometric series
2. To sum a geometric series

1, 2, 4, 8, 16, 32, 64

recursive → Multiply by 2 to get the next term

$\frac{8}{4} = 2 \qquad \frac{64}{32} = 2$

In a geometric sequence, a term is found by multiplying the previous term by a common ratio, r. (so r=2 in the above ex)

So, recursively, $\boxed{a_n = a_{n-1} r}$

common ratio

An Explicit Formula for a_n (geometric sequence)

$a_1 = a_1 r^0$

$a_2 = a_1 r^1$

$$a_2 = a_1 r$$

$$a_3 = a_2 r = (a_1 r) r = a_1 r^2$$

$$a_4 = a_3 r = (a_1 r^2) r = a_1 r^3$$

$$a_5 = a_4 r = a_1 r^{5-1} = a_1 r^4$$

$$\vdots$$

$$\boxed{a_n = a_1 r^{n-1}}$$

Explicit Formula for a Geometric Sequence

ex Find a_5, a_9, and a_n.

$1, -3, 9, -27, \ldots$

a_1

$$r = \frac{-3}{1} = -3$$

$$a_5 = (-27)(-3) = \boxed{81}$$ ← recursive

Explicit

$$a_n = a_1 r^{n-1}$$

$$a_n = 1(-3)^{n-1}$$

$$\boxed{a_n = (-3)^{n-1}}$$

$$a_9 = (-3)^{9-1} = (-3)^8 = 6561$$

Find a formula for the sum of a geometric sequence.

$$\boxed{S_n = \sum_{k=1}^{n} a_k}$$

$$S_n = a_1 + \cancel{a_1 r} + \cancel{a_1 r^2} + \cancel{a_1 r^3} + \cdots + \cancel{a_1 r^{n-1}}$$

$$-rS_n = -\left(\cancel{a_1 r} + \cancel{a_1 r^2} + \cancel{a_1 r^3} + \cdots + \cancel{a_1 r^{n-1}} + a_1 r^n\right)$$

$$1 \cdot S_n - rS_n = a_1 - a_1 r^n$$

$$\frac{S_n\cancel{(1-r)}}{\cancel{1-r}} = \frac{a_1(1-r^n)}{1-r}$$

$$\boxed{S_n = \frac{a_1(1-r^n)}{1-r}}$$

S_n for a geometric series

(ex) Find the sum

a) $\sum_{n=1}^{6} \left(\frac{2}{3}\right)^n = S_6$

don't

a) $\sum_{n=1}^{6} (3)$

$$\boxed{\left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^5 + \left(\frac{2}{3}\right)^6}$$

don't show

a_1

$$r = \frac{2}{3}$$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_6 = \frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^6\right)}{1-\frac{2}{3}}$$

$$= \frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^6\right)}{\frac{1}{3}}$$

$$= 2\left(1-\left(\frac{2}{3}\right)^6\right)$$

$$= \frac{1330}{729}$$

b) $\sum_{n=0}^{7} 2(5)^n$ 8 terms

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$a_1 = 2(5)^0 = 2$

$r = 5$

$n = 8$

$r = 5$

$n = 8$

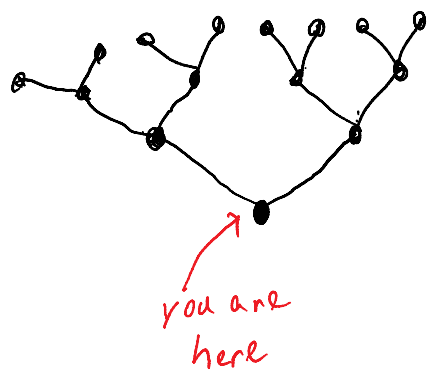
$$S_8 = \frac{2(1-5^8)}{1-5} = \frac{2(1-5^8)}{-4}$$

$$= -\frac{(1-5^8)}{2}$$

$$= 195312$$

ex How many "parents" (parents, grand parents, great grand parents, etc.) do you have going back 10 generations?

$$\underset{1\text{gen}}{\overset{2^1}{2}} + \underset{2\text{gen}}{\overset{2^2}{4}} + \underset{3\text{gen}}{\overset{2^3}{8}} + 16 + 32 + \cdots$$



$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$a_1 = 2$, $r = 2$, $n = 10$

$$S_{10} = \frac{2(1-2^{10})}{1-2}$$

$$= -2(1-2^{10})$$

$$= 2046$$

$= 2046$

$S_{20} = -2(1-2^{20})$

$= 2{,}097{,}150$ people!

$n = 20$ generations

Compound Interest

P = principal (initial amount of \$ invested)

r = yearly interest rate as a decimal

n = number of interest compounding periods per year

t = time in years

A = amount of money earned after time t.

$$\boxed{A = P\left(1+\frac{r}{n}\right)^{nt}}$$

Assume $n=1$

$A_0 = P$

$A_1 = P(1+r)^1$

$A_2 = P(1+r)(1+r) = P(1+r)^2$

$A_3 = P(1+r)^3$

$\vdots$

$A_t = P(1+r)^t$

So, if the compounding of the interest is done n times per year, the compound interest formula is ....

$$A_t = P\left(1+\frac{r}{n}\right)^{nt}$$

Ex. Find the amount of money in an account after 5 years if $12,500 is invested initially with an annual interest rate of 2.5% if the compounding is done (a) semi-annually and (b) quarterly.

a) $\overset{n=2}{A_5} = 12500\left(1+\frac{0.025}{2}\right)^{2\cdot 5} \approx \$14\,153.39$

b) $\overset{n=4}{A_5} = 12500\left(1+\frac{0.025}{4}\right)^{4\cdot 5} \approx \$14\,158.84$

- How to become a millionaire (theoretically)

Invest $5500 per year via a Roth IRA into the stock market.

$n=1$

$$A_t = P(1+r)^t$$

Deposit #	Value of the deposit
1	$5500(1+0.08)^{18}$
2	$5500(1.08)^{17}$
3	$5500(1.08)^{16}$

geometric sequence

2	$5500(1.08)$
3	$5500(1.08)^{1.6}$
⋮	
18	$5500(1.08)^{1} = 5940$

geometric sequence
$r = 1.08$

Amount of money after 18 yrs is the sum of 2nd column

time

Use $S_n = \dfrac{a_1(1-r^n)}{1-r}$ to estimate amount of money after 18 years

$$S_{18} = \frac{5940(1-1.08^{18})}{1-1.08}$$

$$\approx \$222454$$

Sum of an infinite geometric Series

infinite # of terms.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \,?\, 2$$

$S_n = \dfrac{a_1(1-r^n)}{1-r}$, note $r^n \to 0$ as $n \to \infty$ as long as $-1 < r < 1$

$$\lim \frac{a_1(1-\cancel{r^n})}{} \to \frac{a_1}{}, \quad -1 < r < 1$$

$$\lim_{n\to\infty} \frac{a_1(1-\cancel{r^n})}{1-r} \to \frac{a_1}{1-r}, \quad -1<r<1$$

$$\boxed{S = \frac{a_1}{1-r}, \quad -1<r<1}$$

ex Find the sum: a) $1+\frac{1}{2}+\frac{1}{4}+\cdots$

$$r = \frac{1}{2}, \quad a_1 = 1$$

$$S = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

b) $\sum_{n=0}^{\infty} \left(\frac{7}{10}\right)^n$

$$a_1 = \left(\frac{7}{10}\right)^0 = 1, \quad -1<r = \frac{7}{10} < +1$$

$$S = \frac{a_1}{1-r} = \frac{1}{1-\frac{7}{10}}$$

$$= \frac{1}{\frac{3}{10}}$$

$$= \frac{10}{3}$$

ex claim: $0.\overline{9} = 1$

$$0.\overline{9} = 0.99999\ldots$$

$$= 0.9 + 0.09 + 0.009 + 0.0009 + \cdots$$

$-1 < r = 0.1 < 1, \quad a_1 = 0.9$

$$S = \frac{a_1}{1-r} = \frac{0.9}{1-0.1}$$

$$= \frac{0.9}{0.9} = 1$$

ex write as a <u>rational</u> <u>number</u>:

$0.\overline{63}$

$$.\overline{63} = 0.6363636363$$

$$= 0.63 + 0.0063 + 0.000063 + \cdots$$

$r = 0.01, \quad a_1 = 0.63$

$$S = \frac{a_1}{1-r} = \frac{.63}{1-0.01}$$

$$1 - 0.01$$

$$= \frac{.63}{.99}$$

$$= \frac{63}{99}$$

$$= \frac{7}{11}$$

ex $\quad 0.0\overline{123}$