Section 7.4 Part II: Trigonometric Form of Complex Numbers

Thursday, April 10, 2014
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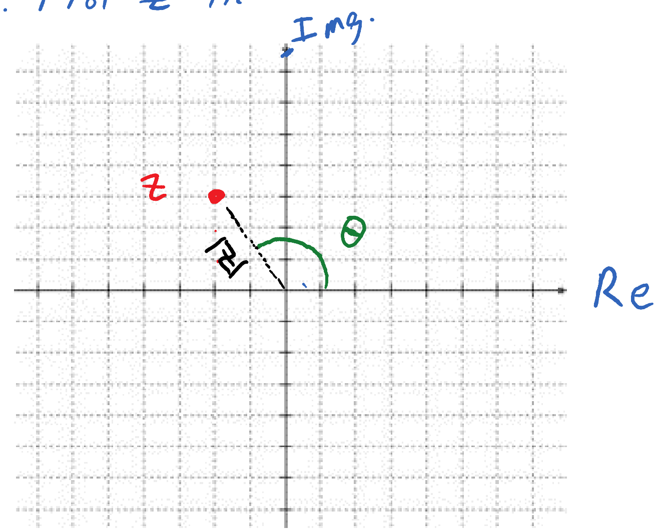
Goal:

1. To plot a complex number in the complex plane.
2. To convert between standard form and trig form.
3. To multiply and divide complex numbers in trig form.

ex) consider $z = -2+3i$. Plot z in complex plane

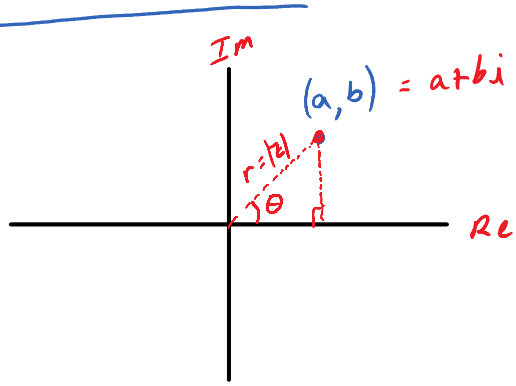


Def of $|z|$

$$\boxed{\text{If } z = a+bi. \text{ Then } r = |z| = \sqrt{a^2+b^2}}$$

$$\tan\theta = \frac{b}{a}$$

Trig form of $z = a+bi$



$$z = a+bi = r\cos\theta + i\, r\sin\theta$$

$$\boxed{z = r(\overset{a}{\cos\theta} + i\overset{b}{\sin\theta})},$$

trig form

for short

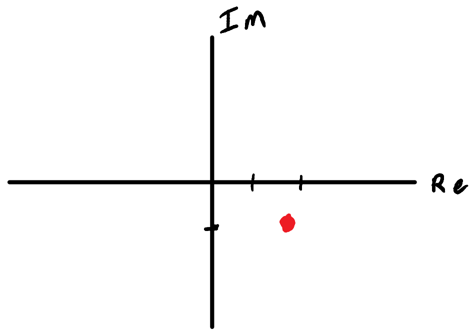
$$\boxed{z = r \operatorname{cis} \theta}$$

θ is the direction angle of z measured from positive real axis.

Relevant Formulas
$r = \sqrt{a^2+b^2}$ ← $\lvert z \rvert$
$\tan\theta = \frac{b}{a}$

ex) convert to trig form

a) $z = \sqrt{3} - 1i$



$$|z| = r = \sqrt{3+1} = \sqrt{4} = 2$$

$$\tan\theta = \frac{-1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$\alpha = 30^0$

$\theta = 330^0$

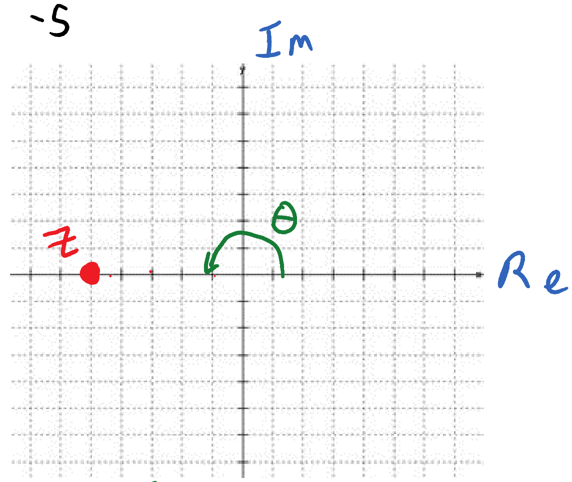
$$z = r(\cos\theta + i\sin\theta)$$

$$= 2(\cos 330^0 + i\sin 330^0)$$

$= 2(\cos 330° + i \sin 330°)$

$= 2 \text{ cis } 330°$

b) $z = -5$



$r = |-5| = 5$

$\theta = 180°$

$z = r \text{ cis } \theta$

$z = 5 \text{ cis } 180°$

Product of Two complex #'s

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

$$= r_1 r_2 \text{ cis}(\theta_1+\theta_2)$$

Proof:

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$

$= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 [\cos\theta_1 \cos\theta_2 + i\sin\theta_2 \cos\theta_1 + i\sin\theta_1 \cos\theta_2 + \overset{(-1)}{i^2} \sin\theta_1 \sin\theta_2]$

$= r_1 r_2 [\underbrace{\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2}_{\cos(\theta_1+\theta_2)} + i(\underbrace{\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1}_{\sin(\theta_1+\theta_2)})]$

$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

\# Done

(ex) multiply

a) $(2 \text{ cis } 30^\circ)(3 \text{ cis } 225^\circ)$

$= 2 \cdot 3 \text{ cis}(30^\circ + 225^\circ)$

$= 6 \text{ cis}(255^\circ)$

b) $5\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) \cdot 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$\frac{2\pi}{3} + \frac{2\pi}{5} = \frac{10\pi}{15} + \frac{6\pi}{15} = \frac{16\pi}{15}$

$= 10 \text{ cis}\left(\frac{16\pi}{15}\right)$

<u>Division</u>

$$\frac{z_1}{z_2} = \frac{r_1(\text{cis}\,\theta_1)}{r_2(\text{cis}\,\theta_2)}$$

$$= \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$$

ex $$\frac{32 \text{ cis } 30^0}{4 \text{ cis } 150^0}$$

$$= 8 \text{ cis}(30 - 150^0)$$

$$= 8 \text{ cis}(-120^\circ)$$

$$= 8 \text{ cis}(240^0)$$

$$z_1 z_2 = r_1 r_2 \left(\cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2)\right)$$
$$= r_1 r_2 \operatorname{cis}(\theta_1+\theta_2)$$

$$z^1 = r \operatorname{cis} \theta$$

$$z^2 = (r r) \operatorname{cis}(\theta+\theta) = r^2 \operatorname{cis} 2\theta$$

$$z^3 = (r^2) \operatorname{cis}(2\theta)(r) \operatorname{cis}(\theta) = r^3 \operatorname{cis}(3\theta)$$

$$\vdots$$

Power

$$\boxed{z^n = r^n \operatorname{cis}(n\theta)}$$

$\leftarrow$ DeMoivre's Theorem for powers

(ex) Evaluate $(1+\sqrt{3}i)^8$

$$z = 1+\sqrt{3}i$$

$$r = \sqrt{1+3} = \sqrt{4} = 2$$

$$\tan\theta = \frac{\sqrt{3}}{1} \rightarrow \theta = 60^\circ$$

$$z = 2 \operatorname{cis} 60^\circ, \quad n = 8$$

$$(2 \operatorname{cis} 60^\circ)^8 = 2^8 \operatorname{cis}(8 \cdot 60^\circ)$$

$$= 256 \operatorname{cis} 480^\circ$$

$$= 256 \operatorname{cis} 120^\circ$$

$$256(\cos 120^\circ + i \sin 120^\circ)$$

$$256\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$-128 + i\,128\sqrt{3}$$

Loose End

$$\operatorname{cis}\theta = \cos\theta + i\sin\theta$$

Turns out $\operatorname{cis}\theta = e^{i\theta}$, where

$e \approx 2.718$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$= -1 + i\,0$$

$$e^{\pi i} = -1$$

$$e^{\pi i} + 1 = 0$$

Euler's Identity

$$\pi = \frac{c}{d}$$

$$i = \sqrt{-1}$$

$$e \approx 2.718$$

1 = multiplicative i.d.

0 = additive identity.