Section 9.4: Partial Fractions

Monday, April 21, 2014
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Goal: To reverse the addition of rational expressions.

ex Add $\frac{-1}{x+4} + \frac{3}{x+8}$ | LCD $=(x+4)(x+8)$

$$= \frac{-1}{(x+4)}\frac{(x+8)}{(x+8)} + \frac{3}{(x+8)}\frac{(x+4)}{(x+4)}$$

$$= \frac{-1(x+8)+3(x+4)}{(x+4)(x+8)}$$

$$= \frac{-x-8+3x+12}{(x+4)(x+8)}$$

$$= \frac{2x+4}{(x+4)(x+8)}$$

ex Find the partial fraction decomposition (p.f.d.) of...

a) (Non-repeated linear factors) ↳ in DEN

$$\frac{2x+4}{x^2+12x+32}$$

① factor DEN and set up decomp.

$$\frac{2x+4}{(x+4)(x+8)} = \underbrace{\frac{A}{x+4} + \frac{B}{x+8}}_{\text{how it decomposes}}$$

② clear DEN's

$$(\cancel{x+4})(\cancel{x+8})\cdot\frac{(2x+4)}{(\cancel{x+4})(\cancel{x+8})} = \frac{A(\cancel{x+4})(x+8)}{(\cancel{x+4})} + \frac{B(x+4)(\cancel{x+8})}{(\cancel{x+8})}$$

$$2x+4 = A(x+8) + B(x+4)$$

③ Multiply out R.H.S, group like terms and factor out the relevant power of x

Shortcut

Aside: works best for non-repeated linear factors

③ Multiply out R.H.S, group like terms and factor out the relevant power of x

$$2x+4 = Ax + 8A + Bx + 4B$$

$$2x+4 = \underbrace{Ax + Bx} + \underbrace{8A + 4B}_{\text{constant}}$$

$$2x+4 = (A+B)x + (8A+4B)$$

④ set corresponding coefficients equal, solve resulting system and fill in decomposition

$$A+B=2 \longrightarrow A = 2-B$$

$$\tfrac{1}{4}(8A+4B=4)$$

$$2A+B=1$$

$$2(2-B)+B=1$$

$$4-2B+B=1$$

$$4-B=1$$

$$B=3 \qquad A=-1$$

$$\frac{A}{x+4} + \frac{B}{x+8}$$

$$\frac{-1}{x+4} + \frac{3}{x+8}$$

Aside: works ^{best} for non-repeated linear factors

$$2x+4 = A(x+8) + B(x+4)$$

$\boxed{\text{Let } x=-8}$	$\boxed{\text{Let } x=-4}$
$-12 = B(-4)$	$-4 = A4$
$B=3$	$A=-1$

b) [Repeated linear factors]

$$\frac{-2x-7}{x^2+4x+4}$$

① $$\frac{-2x-7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

② $$(x+2)^2\left[\frac{(-2x-7)}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}\right]$$

③ $$-2x-7 = A(x+2) + B$$

$$-2x-7 = Ax + \underbrace{2A+B}$$

$$A = -2$$

$$2A + B = -7$$

$$-4 + B = -7$$

$$B = -3$$

$$\frac{-2}{x+2} + \frac{-3}{(x+2)^2}$$

c) (Irreducible Quadratic Factor)

$$\frac{3x^2 + 17x + 14}{x^3 - 8}$$

$$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$$

① $$\frac{3x^2+17x+14}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$

② $$(x-2)(x^2+2x+4)\left[\frac{3x^2+17x+14}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}\right]$$

$$3x^2+17x+14 = A(x^2+2x+4) + (Bx+C)(x-2)$$

③ $$3x^2+17x+14 = Ax^2 + 2Ax + 4A + Bx^2 - 2Bx + Cx - 2C$$

$$3x^2+17x+14 = Ax^2 + Bx^2 + 2Ax - 2Bx + Cx + 4A - 2C$$

$$3x^2+17x+14 = (A+B)x^2 + (2A-2B+C)x + (4A-2C)$$

④ $$A + B = 3$$

$$2A - 2B + C = 17$$

$$4A - 2C = 14$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 2 & -2 & 1 & 17 \\ 4 & 0 & -2 & 14 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 17 \\ 4 & 0 & -2 & 14 \end{array}\right]$$

$\vdots$ RREF

$$\begin{array}{cccc} A & B & C & \\ \multicolumn{4}{c}{\left[\begin{array}{cccc} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]} \end{array} \begin{array}{l} \rightarrow A = 5 \\ \rightarrow B = -2 \\ \rightarrow C = 3 \end{array}$$

$$\frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$

$$\frac{5}{x-2} + \frac{-2x+3}{x^2+2x+4}$$

d) (Repeated Quadratic Factor)

Set up decomp only

$$\frac{3x^3-2x^2+x-2}{(x^2+x+1)^3}$$

$$\frac{3x^3-2x^2+x-2}{(x^2+x+1)^3} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2} + \frac{Ex+F}{(x^2+x+1)^3}$$

$\vdots$ (You try at home)

$A=3, B=-5, C=3, D=3$

$\rightarrow$ If it were $(x^2+x+1)^2$ in DEN

e) $\dfrac{x^3-x^2-x-1}{x^2-x}$ $\Big\}$ improper, so long divide first!

$x^2 - x$) divide first!

$$
\begin{array}{r}
x \\
x^2 - x + 0 \overline{\big) \cancel{x^3} - \cancel{x^2} - x - 1} \\
-\cancel{x^3} + \cancel{x^2} + 0 \\
\hline
-x - 1 \quad R
\end{array}
$$

$$\frac{x^3 - x^2 - x - 1}{x^2 - x} = x + \frac{-x-1}{x^2 - x}$$

Do decomp on this guy

$\vdots$

$$= x + \underbrace{\frac{1}{x} - \frac{2}{x-1}}_{\text{decomposition}}$$

The following is a summary of the kinds of factors you might encounter in the denominator of a rational expression, and how to set up the corresponding decomposition. Note that the rational expression must be "proper" before you decompose it.

Case 1 Nonrepeated Linear Factors

The partial fraction decomposition will contain an expression of the form $\frac{A}{ax + b}$ for each nonrepeated linear factor of the denominator. For example, in the rational expression

$$\frac{3x - 1}{x(3x + 4)(x - 2)}$$

each linear factor of the denominator occurs only once. Thus its partial fraction decomposition has the form

$$\frac{3x - 1}{x(3x + 4)(x - 2)} = \frac{A}{x} + \frac{B}{3x + 4} + \frac{C}{x - 2}$$

Case 2 Repeated Linear Factors

The partial fraction decomposition will contain an expression of the form

$$\frac{A_1}{(ax + b)} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_m}{(ax + b)^m}$$

for each repeated linear factor of multiplicity m. For example, in the rational expression

$$\frac{4x + 5}{(x - 2)^2(2x + 1)}$$

the linear factor $(x - 2)$ is a repeated linear factor. Thus its partial fraction decomposition has the form

$$\frac{4x + 5}{(x - 2)^2(2x + 1)} = \frac{A_1}{x - 2} + \frac{A_2}{(x - 2)^2} + \frac{B}{2x + 1}$$

Aufmann, College Algebra, 8e
http://www.webassign.net/ebooks/aufcolalg8/shell.html?s=a4e2a7bf96b1ca86aa885b1c2f640fcb&c=249283&f=4721409&type=youbook&id=462
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Case 3 Nonrepeated Quadratic Factors

The partial fraction decomposition will contain an expression of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

for each quadratic factor that is irreducible over the real numbers. For example, in the rational expression

$$\frac{x - 4}{(x^2 + x + 1)(x - 4)}$$

the quadratic factor $(x^2 + x + 1)$ is irreducible over the real numbers. Thus its partial fraction decomposition has the form

$$\frac{x - 4}{(x^2 + x + 1)(x - 4)} = \frac{Ax + B}{x^2 + x + 1} + \frac{C}{x - 4}$$

Aufmann, College Algebra, 8e
http://www.webassign.net/ebooks/aufcolalg8/shell.html?s=a4e2a7bf96b1ca86aa885b1c2f640fcb&c=249283&f=4721409&type=youbook&id=4 62
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Case 4 Repeated Quadratic Factors

The partial fraction decomposition will contain an expression of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

for each quadratic factor that is irreducible over the real numbers. For example, in the rational expression

$$\frac{2x}{(x - 2)(x^2 + 4)^2}$$

$(x^2 + 4)$ is a repeated quadratic factor. Thus its partial fraction decomposition has the form

$$\frac{2x}{(x - 2)(x^2 + 4)^2} = \frac{A_1x + B_1}{x^2 + 4} + \frac{A_2x + B_2}{(x^2 + 4)^2} + \frac{C}{x - 2}$$

Aufmann, College Algebra, 8e
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