Representing Functions as Power Series

Goal: To express functions as sums of power series

Recall: ① Power series: $\sum_{n=0}^{\infty} c_n (x-a)^n$

$= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$

② Geometric Series: $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

$= \frac{1}{1-x}, \quad |x| < 1$

$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}}$

ex Find a power series for...

$\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad -1 < r < 1}$

a) $f(x) = \frac{4}{x+2}$

$= 4 \cdot \frac{1}{2+x}$

$= \frac{4}{2} \cdot \frac{1}{1+\frac{x}{2}}$

$= 2 \cdot \frac{1}{1-\left(-\frac{x}{2}\right)}$ $\quad r = -\frac{x}{2}$

$= 2 \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$

$= \boxed{2 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}}, \quad -1 < -\frac{x}{2} < 1$

$2 > x > -2$

$-2 < x < 2$

$\boxed{I = (-2, 2) \qquad R = 2}$

b) $f(x) = \frac{1}{x}$ $\qquad \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad -1<r<1$

$= \frac{1}{1-\underbrace{(1-x)}_{r}}, \quad r = 1-x$

$= \sum_{n=0}^{\infty} (1-x)^n$

$= \sum_{n=0}^{\infty} [-1(x-1)]^n$

$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n$ $\qquad \sum_{n=0}^{\infty} c_n (x-a)^n$

$-1 < 1-x < 1$

$-2 < -x < 0$

$2 > x > 0$

$0 < x < 2$

$I = (0, 2), \quad R = 1$

Theorem: If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence $R > 0$, then on the interval $\underbrace{(a-R, a+R)}_{I}$ f is continuous and ...

x^n

$n x^{n-1}$

① $f'(x) = \sum_{n=1}^{\infty} c_n n (x-a)^{n-1}$

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$

$f'(x) = 0 + c_1 + 2c_2(x-1) + \cdots$

② $\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1} + C$

Notes: ① The radius of convergences, R, remains the same for $f(x)$, $\int f(x)\,dx$, and $f'(x)$.

② The interval of convergence may differ for $f(x)$, $\int f(x)\,dx$, and $f'(x)$ at the endpoint(s)

ex Find a power series centered at 1 for $f(x) = \ln x$, and its interval of convergence.

From last ex, we know $\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$, for $0 < x < 2$. Thus on the interval $(0,2)$,

$$\boxed{\ln x} = \int \frac{1}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n\,dx$$

$$= \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}}$$

We can find C by setting $x = 1$.

$$\ln(1) = C + \overset{\text{sum}}{0} = 0 \quad \ln(1) \text{ (previous knowledge)}$$

$$C = 0$$

$$\boxed{\ln x = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}}_{I = (0,2]} \quad \text{on } (0,2)}$$

check (use AST)

ex use Power Series to approximate within six decimal places.

$|\text{error}| < 0.000001$

$$\int_0^{0.3} \frac{x^2}{1+x^4}\,dx$$

$$\boxed{\sum_{n=0}^{\infty} r^n = \boxed{\frac{1}{1-r}}}$$

Time-out

$\frac{x^2}{1+x^4}$ find a power series

$x^2 \cdot \boxed{\frac{1}{1-(-x^4)}}$ $\qquad \frac{1}{1-r}$

$$x^2 \sum_{n=0}^{\infty} (-x^4)^n$$

$x^2 \sum_{n=0}^{\infty} (-1)^n (x^4)^n$

$x^2 \sum_{n=0}^{\infty} (-1)^n x^{4n}$

$\sum_{n=0}^{\infty} (-1)^n x^{4n+2}$

$\boxed{x^2 \cdot x^{4n} \quad x^{4n+2}}$

$-1 < -x^4 < 1$

$-1 < x^4 < 1$

$-1 < x < 1$

Time-in

$\int_0^{0.3} \sum_{n=0}^{\infty} (-1)^n x^{4n+2} \, dx \quad , \quad -1 < x < 1$

$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{4n+3} \right]_0^{0.3}$

$\sum_{n=0}^{\infty} \frac{(-1)^n (0.3)^{4n+3}}{4n+3} \quad$ for $\quad -1 < x < 1$

use ASET

$\underbrace{\frac{(0.3)^3}{3} - \frac{(0.3)^7}{7} + \frac{(0.3)^{11}}{11}} - \frac{0.3^{15}}{15}$ < Error

0.008968