The Comparison Tests

Goal: To determine series convergence using the Comparison Test and the Limit Comparison Test.

Comparison Test

Let $0 \le a_n \le b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

ex Does it converge?

$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p}}$ p-series

a) $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$

1. $0 < \frac{1}{3n^2+2} < \frac{1}{n^2}$
2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p=2>1$)
3. So $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ converges by the Comparison Test.

b) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

1. $0 < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}-1}$
2. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ diverges (p-series, $p=\frac{1}{2}\le 1$)
3. So, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges the comparison theorem.

c) $\sum_{n=0}^{\infty} \frac{2^n}{3^n+1}$

1. $0 < \frac{2^n}{3^n+1} < \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$
2. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ converges (geometric, $-1<r=\frac{2}{3}<1$)

Limit Comparison Test: Let $a_n, b_n > 0$.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ (L finite) and $L>0$.

Then either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or both diverge.

Indication: $\frac{a_n}{b_n} \to L$, So $\frac{a_n}{b_n} \approx L$ (for "large" n)

Thus $a_n \approx L b_n$ for "large" n.

Say, $\sum b_n$ converges. Then $L \sum b_n = \sum L b_n$ converges. But $\sum a_n \approx \sum L b_n$. So $\sum a_n$ also converges.

becomes "=" as $n \to \infty$.

ex) Does it converge?

a) $\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$

Trick for find b_n by taking in ratio the highest power n in NUM of a_n to " " " " DEN " "

① $b_n = \frac{n}{n^2} = \frac{1}{n} > 0$.

$$\frac{a_n}{b_n} = \frac{\frac{5n-3}{n^2-2n+5}}{1/n} = \frac{(5n-3)}{n^2-2n+5} \cdot n = \frac{5n^2-3n}{n^2-2n+5} \longrightarrow \frac{5}{1} = 5 > 0$$

So LCT applies.

② $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series, $p = 1 \le 1$)

③ $\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$ diverges by LCT.

b) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^6+1}}$

① $b_n = \frac{1}{n\sqrt{n^6}} = \frac{1}{n \cdot n^3} = \boxed{\frac{1}{n^4}} > 0$.

$$\frac{a_n}{b_n} = \frac{1}{n\sqrt{n^6+1}} \cdot \frac{n^4}{1} = \frac{n^{\cancel{4}\,3}}{\cancel{n}\sqrt{n^6+1}}.$$

$$\lim_{n\to\infty} \frac{n^3}{\sqrt{(n^6+1)}} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{1}{\sqrt{(n^6+1)\frac{1}{n^6}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\cancel{\frac{1}{n^6}}^{\,0}}} = \frac{1}{\sqrt{1}} = 1 > 0.$$

So, LCT applies.

② $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges (p-series, $p = 4 > 1$).

③ So, $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^6+1}}$ converges by LCT