Section 13.3: Arc Length and Curvature

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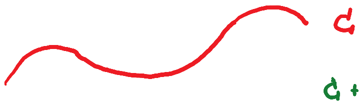
Goals:
1. To find the arc length of a curve
2. To calculate an arc length parameterization of C.
3. To calculate K.

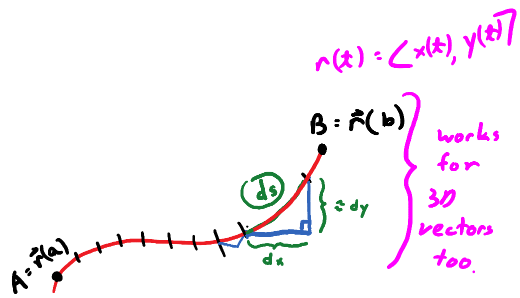
Convention: C is a smooth curve given by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$



Theorem: Arc Length, s, of C $\underbrace{\text{from } t=a \text{ to } t=b}_{}$... ($C$ traced out once)

$$\boxed{s = \int_a^b \|\vec{r}\,'(t)\|\,dt}$$

Intuitive Justification:



$$(ds)^2 = (dx)^2 + (dy)^2$$

$$ds = \sqrt{(dx)^2 + (dy)^2} \cdot \frac{dt}{dt}$$

$$ds \approx \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}_{\underbrace{\sqrt{(x'(t))^2 + (y'(t))^2}}_{\|\vec{r}\,'(t)\|}\,dt}$$

Form a Riemann sum involving the lengths of all such sub-arcs, take limit as the number of sub-arcs $\to \infty$ to get

$$\boxed{s = \int_a^b \|\vec{r}\,'(t)\|\,dt}$$

(ex) Find the arc length of

$\vec{r}(t) = \langle 3t, -2\sin t, 2\cos t \rangle$ from $t = 0$ to $t = \frac{\pi}{2}$.

$0 \uparrow a$, $\frac{\pi}{2} \uparrow b$

$$s = \int_a^b \|\vec{r}\,'(t)\|\,dt$$

$$\boxed{s = \int_a^b \|\vec{r}\,'(t)\|\,dt}$$

$$\vec{r}\,'(t) = \langle 3, -2\cos t, -2\sin t \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{9 + 4\cos^2 t + 4\sin^2 t}$$

$$= \sqrt{9 + 4(\underbrace{\cos^2 t + \sin^2 t}_{1})}$$

$$= \sqrt{13}$$

$$s = \int_0^{\frac{\pi}{2}} \sqrt{13}\,dt$$

$$= \sqrt{13}\int_0^{\frac{\pi}{2}} 1\,dt$$

$$= \sqrt{13}\,\Big[t\Big]_0^{\frac{\pi}{2}}$$

$$= \sqrt{13}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi\sqrt{13}}{2} \text{ units}$$

Def: Arc Length Function

$$\boxed{s(t) = \int_a^t \|\vec{r}\,'(u)\|\,du}$$

Notes

① $s(t)$ meaures distance along C from $\vec{r}(a) = \langle x(a), y(a), z(a)\rangle$ to $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

② s (from $s(t)$) is called the arc length parameter

③ $\dfrac{ds}{dt} = \|\vec{r}\,'(t)\|$

ex Find an arc length parameterization of

$\vec{r}(t) = (3+2t)\vec{i} + (4+t)\vec{j} - 5t\vec{k}$ with $a=0$.

(want $\vec{r}$ in terms of s).

$$\vec{r}\,'(u) = 2\vec{i} + 1\vec{j} - 5\vec{k}$$

$$\|\vec{r}\,'(u)\| = \sqrt{4+1+25} = \sqrt{30}$$

$$s(t) = \int^t \|\vec{r}\,'(u)\|\,du$$

$\vec{r}\,'(u) = 2\vec{i} + 1\vec{j} - 5\vec{k}$

$\|\vec{r}\,'(u)\| = \sqrt{4+1+25} = \sqrt{30}$

$s(t) = \int_a^t \|\vec{r}\,'(u)\|\,du$

$\underbrace{s(t)} = \int_0^t \sqrt{30}\,du$

$s = \sqrt{30}\int_0^t 1\,du$

$s = \sqrt{30}\,t$

$\underbrace{t = \frac{s}{\sqrt{30}}}$

$t(s) = \frac{s}{\sqrt{30}}$

$\underbrace{\vec{r}\,[t(s)]}_{\vec{r}(s)} = \left(3 + \frac{2s}{\sqrt{30}}\right)\vec{i} + \left(4 + \frac{s}{\sqrt{30}}\right)\vec{j} - \frac{5s}{\sqrt{30}}\vec{k}$

Note: when $s=9$, $\vec{r}\,[t(9)]$ is the point on C_1 that 9 units along C_1 from the point $\vec{r}(0)$.

Notes: ① $\vec{r}\,'(s) = \vec{T}(t)$, unit tangent vector

Proof: $\boxed{\vec{T}(t)} = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} \overset{*}{=} \frac{d\vec{r}}{\cancel{dt}} \cdot \frac{\cancel{dt}}{ds}$ } $s(t)$ must be differentiable and invertible for $*$ to be true

$\overset{\text{chain rule}}{=} \frac{d\vec{r}}{ds}$

$= \boxed{\vec{r}\,'(s)}$

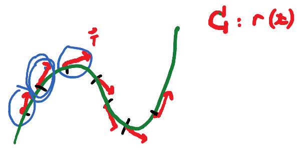
② If $\|\vec{r}\,'(m)\| = 1$, then m is the arc length parameter $(m = s)$. } So, s uniquely parameterizes C_1

Def: $(\text{curvature of } C : \vec{r}(s)) = K = \left\|\frac{d\vec{T}}{ds}\right\| = \|\vec{T}\,'(s)\|$

Note: K is the magnitude of the rate of change of unit tangent vector w.r.t. s. Since $\|\vec{T}\|=1$, the only change that occurs in $\vec{T}$ along C_1 is due to the bend in C. (more Bend) = (larger K).

$C : r(t)$

bend in C. (more bend) → (larger κ).



Theorem: $$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

Lemma: $\vec{T} \cdot \vec{T}' = 0$

Proof: $\|\vec{T}\|^2 = \boxed{\vec{T} \cdot \vec{T} = 1}$

$$\frac{d}{dt}(\vec{T} \cdot \vec{T}) = \frac{d}{dt} 1$$

$$\vec{T}' \cdot \vec{T} + \vec{T} \cdot \vec{T}' = 0$$

$$2\vec{T}' \cdot \vec{T} = 0$$

$$\vec{T}' \cdot \vec{T} = 0$$

Done

Theorem: $$\boxed{\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}}$$

Pf: Involves above lemma

and $\vec{r}'(t) = \|\vec{r}'(t)\| \, \vec{T}(t)$

find $\vec{r}'$ and $\vec{r}''$.

(ex) Find $\kappa(t)$ for $\vec{r}(t) = 2t^2\vec{i} + t\vec{j} + \frac{1}{2}t^2\vec{k}$

$$\vec{r}'(t) = 4t\vec{i} + 1\vec{j} + t\vec{k}$$

$$\vec{r}''(t) = 4\vec{i} + \vec{k}$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4t & 1 & t \\ 4 & 0 & 1 \end{vmatrix} = \vec{i} - (4t - 4t)\vec{j} - 4\vec{k}$$

$$= 1\vec{i} - 4\vec{k}$$

$$\|\vec{r}' \times \vec{r}''\| = \sqrt{1 + 16} = \sqrt{17}$$

$$\|\vec{r}'\|^3 = \left(\sqrt{16t^2 + 1 + t^2}\right)^3 = \left(\sqrt{17t^2 + 1}\right)^3$$

$$\kappa(t) = \frac{\sqrt{17}}{\left(\sqrt{17t^2+1}\right)^3} = \frac{\sqrt{17}}{(17t^2+1)^{\frac{3}{2}}}$$

Note: To find κ for $y = f(x)$. Let $x = t$

$\vec{r}(t) = \langle t, f(t), 0\rangle$

ex Let $f(x) = 2x^2 + 5$. Find k at $x = -1$

let $x = t$, $\vec{r}(t) = \langle t, 2t^2+5, 0\rangle$

$\vec{r}\,'(t) = \langle 1, 4t, 0\rangle$

$\vec{r}\,''(t) = \langle 0, 4, 0\rangle$

$\vec{r}\,' \times \vec{r}\,'' = 4\vec{k}$

$\|\vec{r}\,' \times \vec{r}\,''\| = 4$

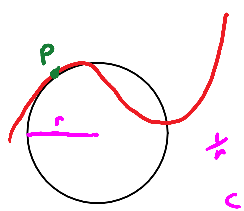
$\|\vec{r}\,'\|^3 = \left(\sqrt{16t^2+1}\right)^3$

$$k(t) = \frac{4}{\left(\sqrt{16t^2+1}\right)^3}$$

$$k(-1) = \frac{4}{(\sqrt{17})^3} = \frac{4}{17^{3/2}}$$

Notes: ① A circle has constant $k = \frac{1}{r}$

②



$\frac{1}{r}$ estimates k at P.

circle is called the circle of curvature to C at P.

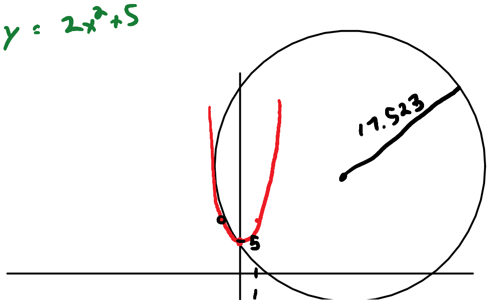
i.e. if C has curvature k at point P, then the circle passing through P w/ $r = \frac{1}{k}$ is called the circle of curvature as long as the circle lies on the concave side of the curve and shares a common tangent with the curve at P.

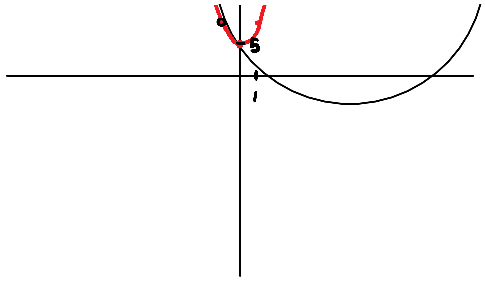
ex Find the radius of the circle of curvature of $\boxed{y = 2x^2 + 5}$ at $x = -1$

$$k = \frac{4}{17^{3/2}}$$

$$r = \frac{17^{3/2}}{4} \approx 17.523$$

$y = 2x^2 + 5$





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