Tangent Planes and Linear Approximations

Goals:

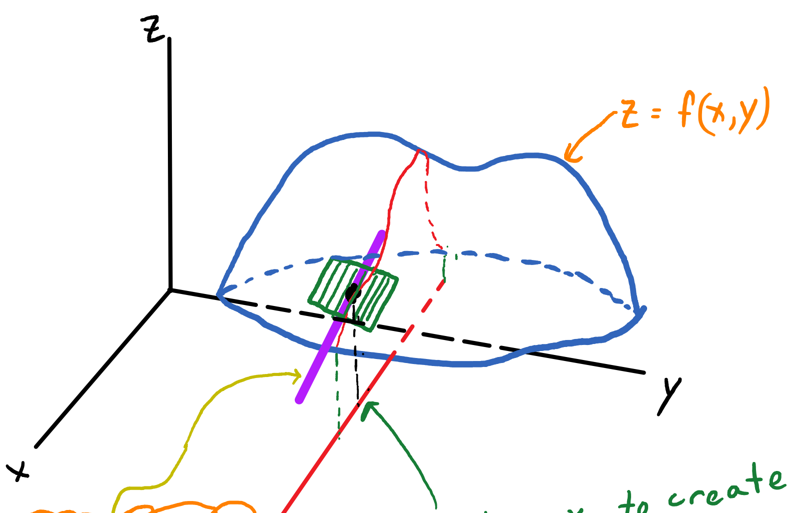
1. To find the tangent plane to a surface at a given point.
2. To calculate the total differential of a multivariable function.
3. To determine whether or not a function is differentiable.

Derivation of the Tangent Plane to a Surface at a Point

Note that the tangent plane to $z = f(x,y)$ at (x_0, y_0, z_0) has equation of the form $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

or $\quad z - z_0 = a(x-x_0) + b(y-y_0)$.

solve for this



hold $y = y_0$ to create trace in red.

Purple tangent line has equation $z - z_0 = a(x - x_0)$ in xz-plane. The slope is $a = f_x(x_0, y_0)$

similarly, if you hold $x = x_0$, you'll get $b = f_y(x_0, y_0)$.

So the equation of the tangent plane is ...

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

ex a) Find the equation of the tangent plane to $f(x,y) = x^2 - 2xy + y^2$ at $(1,2,1)$.

$f_x(x,y) = 2x - 2y \longrightarrow f_x(1,2) = 2 - 4 = -2$

$f_y(x,y) = -2x + 2y \longrightarrow f_y(1,2) = -2 + 4 = 2$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$\boxed{z - 1 = -2(x - 1) + 2(y - 2)}$$

b) Find a linearization, $\underbrace{L(x,y)}_{z \text{ from}}$, for $f(x,y)$ at $(1,2,1)$.

$$L(x,y) = -2(x-1) + 2(y-2) + 1$$

$$= -2x + 2 + 2y - 4 + 1$$

$$= -2x + 2y - 1$$

c) Use $L(x,y)$ to estimate $f(0.95, 2.02)$

$$L(0.95, 2.02) = -2(0.95) + 2(2.02) - 1$$

$$= 1.14$$

[Turns out $f(0.95, 2.02) \approx 1.1449$]

ex Consider the tangent plane equation:

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0 .$$

Let $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$ and discuss what the result represents geometrically.

$$z = f_x(x_0,y_0)\big((x_0+\Delta x)-x_0\big) + f(x_0,y_0)\big((y_0+\Delta y)-y_0\big) + z_0$$

$$= \underbrace{f_x(x_0,y_0)\,\Delta x + f(x_0,y_0)\,\Delta y}_{\text{change in height of the tangent plane from inputs of } (x_0,y_0) \text{ to inputs of } (x_0+\Delta x,\, y_0+\Delta y)} + z_0$$

Definitions

(1) $$\underbrace{dz}_{\text{total differential}} = f_x(x,y)\overbrace{dx}^{dx=\Delta x} + f_y(x,y)\overbrace{dy}^{dy=\Delta y}$$

Recall ① from calc I: $\underbrace{\overbrace{dy = f'(x)dx}^{\text{for a particular } x_0 \text{, this is the change in tangent line height from } x_0 \text{ to } x_0+\Delta x.}}_{\text{differential for } y=f(x)}$

(2) $\Delta y \approx dy$ where $\Delta y = f(x_0+\Delta x) - f(x)$, f differentiable and Δx "small."

$[\Delta y = \text{change in height of } y=f(x_0) \text{ from } x_0 \text{ to } x_0+\Delta x]$

② $\underbrace{\Delta z}_{\text{change in height of } f \text{ from } (x_0,y_0) \text{ to } (x_0+\Delta x,\, y_0+\Delta y)} = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$

Note: For "well-behaved" functions and for "small" $\Delta x, \Delta y$, $dz \approx \Delta z$ when going from inputs (x_0, y_0) to $(x_0+\Delta x, y_0+\Delta y)$.

(ex) Let $z = \sin(2x+3y)$. Use dz to estimate Δz from $(-3,2)$ to $(-2.95, 2.04)$

$$dz = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$

$$f_x(x,y) = 2\cos(2x+3y) \longrightarrow f_x(-3,2) = 2$$
$$f_y(x,y) = 3\cos(2x+2y) \longrightarrow f_y(-3,2) = 3$$

$$dz = 2(0.05) + 3(0.04) = \boxed{0.22}$$

Note: $\Delta z \approx 0.2182$

Note: ① In calc I, differentiability $\Rightarrow$ continuity.

② **Problem**: paritials exist $\not\Rightarrow$ f is continuous

see (ex) below

[we want a definition of differentiability for $z = f(x,y)$ such that f differentiable $\Rightarrow$ f is continuous.]

③ solution: come up with a definition of a differentiable function that ensures $\Delta z \approx dz$ for "small" input changes.

Definition: $z = f(x,y)$ is differentiable at (x_0, y_0) as long as Δz can be written as...

$$\Delta z = \underbrace{f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y}_{dz|_{(x_0,y_0)}} + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Theorem 1: $z = f(x,y)$ differentiable at (x_0,y_0)

$\Rightarrow$ f is continuous (x_0,y_0).

Proof: uses the above def to show $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

means f is continuous at (x_0,y_0)

Theorem 2: f_x, f_y continuous on D, then f is differentiable on D.

Partials exist but f not differentiable at $(0,0)$!

ex Let $f(x,y) = \begin{cases} \dfrac{2xy}{2x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Find the partials $f_x(0,0)$ and $f_y(0,0)$, if they exist.

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} = \frac{2\,\Delta x \cdot 0}{2(\Delta x)^3} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, 0+\Delta y) - f(0,0)}{\Delta y} = \frac{2 \cdot 0 \cdot \Delta y}{(\Delta y)^3} = 0$$

But $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ DNE! So, f is not continuous at $(0,0)$ but its partials exist.

we showed this in 14.2

[ie. partials exist $\not\Rightarrow$ f differentiable].

So, just because the partials exist at a point does not mean that the function is well-behaved at that point.

f behaving badly
around input (0,0)