Section 14.7: Extrema of Two Variable Functions

Thursday, March 5, 2015 3:33 PM

Goal:

1. To find the absolute extrema of a two-input variable function.
2. To find local (or relative) extrema using the Second Derivative Test.

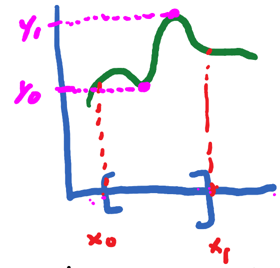
closed region R in $\mathbb{R}^2$

Definitions: 1. Suppose a function f is defined on a closed bounded region R with $f(x_0,y_0) \le f(x,y) \le f(x_1,y_1)$ for all (x,y) in R. Then $f(x_0,y_0)$ and $f(x_1,y_1)$ are called the **absolute minimum** and **absolute maximum** of f in R.

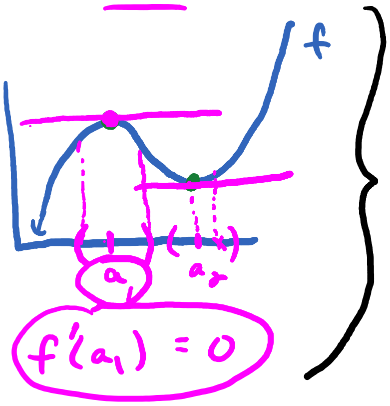
relative = local

2. A function of two variables has a **local maximum** at (a,b) if $f(x,y) \le f(a,b)$ for all points (x,y) in an open disk with center (a,b).
3. A function of two variables has a **local minimum** at (a,b) if $f(x,y) \ge f(a,b)$ for all points (x,y) in an open disk with center (a,b).
4. A point (a,b) is called a **critical point** of f is $f_x(a,b) = 0$ and $f_y(a,b) = 0$ or if one of these partial derivatives does not exist.

open disk in $\mathbb{R}^2$



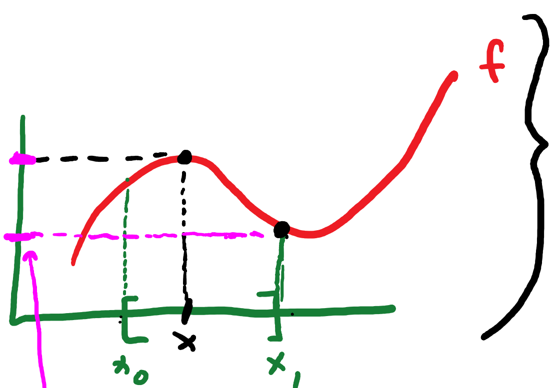
closed interval $[x_0, x_1]$ is calc I version of a closed region.



calc I version of critical values, a_1 and a_2, and local extrema occuring at critical values

Theorem 1: **(Extreme Value Theorem)** Suppose f is a continuous function of two variables on a closed bounded region R in the xy-plane. Then f takes on both an absolute maximum and absolute minimum in R.

$f(x,y)$



abs max on closed interval $[x_0, x_1]$

abs. min on $[x_0, y_0]$

calc I pic illustrating E.V.T.

Theorem 2: If f has a local maximum or minimum at (a,b) and the first-order partial derivatives of f exist there, then $f_x(a,b)=0$ and $f_y(a,b)=0$.

Note:

1. Although a critical point is not necessarily a minimum or maximum, local extrema occur only at critical points.
2. The critical points of a function f can also be defined as the points in the domain of f for which $\nabla f(x,y)=\mathbf{0}$ or is undefined. Filling in 0 for the partial derivatives in the tangent plane formula from 15.4 (14.4) gives the equation $z=z_0$, a horizontal plane. Thus, the critical values of a function correspond to the points where the function's tangent plane is horizontal or does not exist.

$\nabla f = f_x \hat{i} + f_y \hat{j} = \langle \underset{=0}{f_x}, \underset{=0}{f_y} \rangle$

$$\boxed{z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

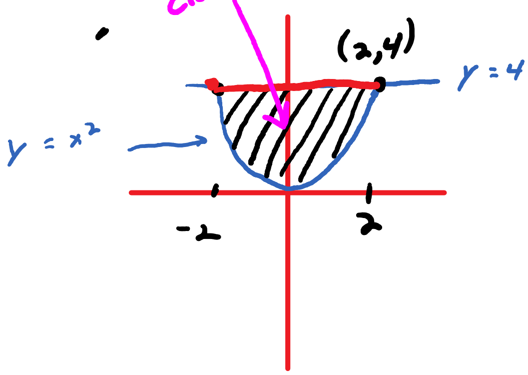
$z - z_0 = 0$ when $\nabla f = 0$

$z = z_0$ horizontal tangent plane

ex Let $f(x,y) = 3x^2 + 2y^2 - 4y$. Find the absolute extrema over the region $\underbrace{y = x^2 \text{ and } y = 4}$

bounded by graphs, a closed bounded region.

closed, bounded region



by EVT, there will abs extrema on this closed, bounded region since f is a polynomial (all polys are continuous)

① Interior c.p.'s

$f(x,y) = 3x^2 + 2y^2 - 4y$

$$f_x(x,y) = \boxed{6x = 0} \to x = 0$$

$$f_y(x,y) = \boxed{4y - 4 = 0} \to y = 1$$

c.p. $(0,1)$

② Boundary $y = 4$, $-2 \le x \le 2$

$$f(x,4) = 3x^2 + 2(4)^2 - 4(4)$$

$$f(x,4) = 3x^2 + 16$$

c.p. $(0,4)$

Also check $x = -2$, $x = 2$

$(-2,4)$ $(2,4)$

③ Boundary : $y = x^2$ $\quad f(x,y) = 3x^2 + 2y^2 - 4y$

$$f(x, x^2) = 3x^2 + 2(x^2)^2 - 4 \cdot x^2$$

$$g(x) = f(x, x^2) = 2x^4 - x^2$$

$$g(x) = 2x^4 - x^2$$

$$g'(x) = \boxed{8x^3 - 2x = 0}$$

$$2x(4x^2 - 1) = 0$$

$2x = 0$ or $4x^2 - 1 = 0$

$\boxed{\begin{array}{l} x = 0 \\ y = 0 \end{array}}$ $\quad x^2 = \frac{1}{4}$

$x = \pm \frac{1}{2}$

$(0,0)$ $\left(\pm \frac{1}{2}, \frac{1}{4}\right)$

Use a table to find the abs. extrema

x	y	$f(x,y) = 3x^2+2y^2-4y$
0	1	-2 abs. min
0	4	16
-2	4	28 abs. max
2	4	28
0	0	0
$1/2$	$1/4$	$-1/8$
$-\frac{1}{2}$	$+\frac{1}{4}$	$-\frac{1}{8}$

$2-4=-2$

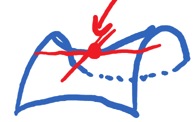
Theorem 3: (**Second Derivatives Test**) Suppose the second partials of f are continuous on an open region containing (a,b) and $\nabla f(a,b)=\mathbf{0}$ (i.e. (a,b) is a critical point). Let

$$d = f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$$

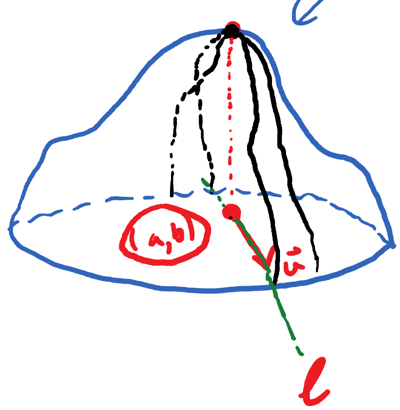
$\boxed{f''(a)<0}$ similar calc I result indicating local max corresponding to $x=a$ where a is a critical value

a) If $d>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.
b) If $d>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.
c) If $d<0$, then $f(a,b)$ is neither a local minimum nor a local maximum.
d) The test is inconclusive if $d=0$.

saddle pt.



$z=f(x,y)$



To show part (b) is true,
Take 2nd order directional derivative in the direction of a generic $\vec{u}$. Show trace is concave down in an open disk containing (a,b) (for any such $\vec{u}$). Then $f(a,b)$ is a local max by calc I 2nd derivative test.

(ex) Find the local extrema

$$\rightarrow f(x,y) = x + y + \frac{1}{xy}, \quad x, y > 0$$

$$f_x(x,y) = \boxed{1 - \frac{1}{x^2 y} = 0}$$

$$f_y(x,y) = \boxed{1 - \frac{1}{x y^2} = 0}$$

$$\frac{d}{dx}\left(\frac{x^{-1}}{y}\right) = -\frac{x^{-2}}{y}$$

$$\frac{1}{x^2 y} = 1$$

$$\frac{1}{x y^2} = 1$$

$$\frac{1}{x^2 y} = \frac{1}{x y^2}$$

$$x^2 y = x y^2$$

$$x^2 y - x y^2 = 0$$

$$xy(x - y) = 0$$

$$x - y = 0$$

$$y = x$$

$$\frac{1}{x^3} = 1$$

$$x^3 = 1$$

$$x = 1$$

$$y = 1$$

$(1,1)$ c.p.

$$d = \overbrace{f_{xx}(1,1)}^{2}\,\overbrace{f_{yy}(1,1)}^{2} - \left(\overbrace{f_{xy}(1,1)}^{1}\right)^2$$

$$f_x(x,y) = \boxed{1 - \frac{1}{x^2 y} = 0}$$

$$f_y(x,y) = \boxed{1 - \frac{1}{x y^2} = 0}$$

$$f_{xx}(x,y) = \frac{2}{x^3 y} \xrightarrow[\text{at }(1,1)]{\text{eval}} 2 > 0$$

$$f_{yy} = \frac{2}{x y^3} \rightarrow 2$$

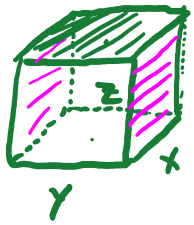
$$f_{xy} = \frac{1}{x^2 y^2} \rightarrow 1$$

$d = 4 - 1 = 3 > 0$

$\boxed{f_{xx}(1,1) = 2 > 0}$, which means $f(1,1) = 3$ is a <u>local minimum</u>

[no local max since $f \to \infty$ as $x, y \to \infty$ or $x, y \to 0$]

ex Find the dimensions of the box with largest volume if the total surface area is 64 cm^2.



$\boxed{V = xyz}$

$A = \boxed{2xz + 2yz + 2xy = 64}$

$xz + yz + xy = 32$

$(x+y)z = 32 - xy$

$z = \frac{32 - xy}{x+y}$

$V(x,y) = xy\left(\frac{32 - xy}{x+y}\right)$

$V(x,y) = \frac{32xy - x^2y^2}{x+y}$

$V_x = \frac{(x+y)(32y - 2xy^2) - (32xy - x^2y^2)\cdot 1}{(x+y)^2}$

$$= \frac{3\cancel{2}xy - \cancel{2x^2y^2} + 32y^2 - 2xy^3 - 3\cancel{2}xy + \cancel{1x^2}y^2}{(x+y)^2}$$

$$= \frac{-x^2y^2 + 32y^2 - 2xy^3}{(x+y)^2}$$

$$\boxed{\begin{aligned} V_x &= \frac{y^2(-x^2+32-2xy)}{(x+y)^2} = 0 \\ V_y &= \frac{x^2(-y^2+32-2xy)}{(x+y)^2} = 0 \end{aligned}}$$

$$-x^2+32-2xy = 0 \longrightarrow -x^2+32-2x^2 = 0$$

$$-(-y^2+32-2xy = 0)$$

$$-3x^2 = -32$$

$$x^2 = \frac{32}{3}$$

$$-x^2+y^2 = 0$$

$$y^2 = x^2$$

$$x = \sqrt{\frac{32}{3}} = y$$

$$y = x$$

$$z = \frac{32 - xy}{x+y}$$

$$z = \frac{\frac{32}{1} - \frac{32}{3}}{\sqrt{\frac{32}{3}} + \sqrt{\frac{32}{3}}}$$

$$z = \frac{\frac{64}{3}}{2\sqrt{\frac{32}{3}}} = \frac{\overset{32}{64}}{3\cdot 2}\cdot\frac{\sqrt{3}}{\sqrt{32}}$$

$$= \frac{32}{3}\cdot\frac{\sqrt{3}}{\sqrt{32}}$$

$$\overset{8}{32} \quad \sqrt{3}$$

$$3 \qquad \sqrt{32}$$

$$= \frac{\cancel{32}^{\,8}}{3} \cdot \frac{\sqrt{3}}{\cancel{4}\sqrt{2}}$$

$$= \frac{8\sqrt{3}}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\cancel{8}^{\,4}\sqrt{6}}{3\cdot\cancel{2}}$$

$$= \frac{4\sqrt{6}}{3} = \sqrt{\frac{32}{3}}$$

$$x = y = z$$

The dimensions are $\sqrt{\frac{32}{3}}$ cm $\times \sqrt{\frac{32}{3}}$ cm $\times \sqrt{\frac{32}{3}}$ cm