Section 14.8: Lagrange Multipliers

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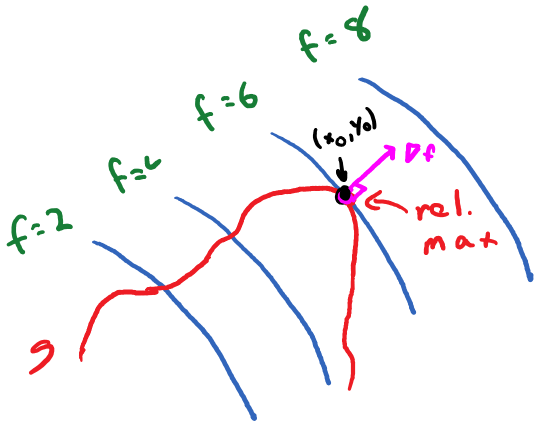
Goal: To solve constrained optimization problems using Lagrange Multipliers



$z = f(x,y)$ only allowed to plug in (x,y) values from $\underbrace{g(x,y) = c}_{\text{a curve in } x,y \text{ plane}}$

level curve of g

Assume (x_0,y_0) is on $\overbrace{g(x,y)=c}^{\text{curve}}$ and it produces a local max on f under restricted domain g. Note that both $\nabla f(x_0,y_0)$ and $\nabla g(x_0,y_0)$ are normal to level curve of f at (x_0,y_0) (since g is tangent to f's level curve at (x_0,y_0)). So $\nabla f \parallel \nabla g$ at $(x_0,y_0) \Longrightarrow \boxed{\nabla f = \lambda \nabla g}$ ✱ at (x_0,y_0).

this leads to the method of Lagrange multipliers for finding extrema

✱ $\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$

Note: ① λ is called a Lagrange multiplier

② ✱ also holds if (x_0,y_0) produces a minimum

Note: ① λ is called a Lagrange multiplier

② $\star$ also holds if (x_0, y_0) produces a minimum value on f.

③ On the constraint, g, the extrema are absolute.

④ As always, f is differentiable, g is smooth.

⑤ $\nabla g(x_0, y_0) \neq \vec{0}$ (It could turn out that $\nabla f(x_0, y_0) = \vec{0}$ when $\lambda = 0$)

Method

① Find all values of x, y, and λ such that $\nabla f = \lambda \nabla g$

② Evaluate f at pts from ① to find the abs. extrema of f on the constraint.

ex) Find the extrema of $f(x,y) = 2x^2 + x + y^2 - 2$

on $x^2 + y^2 = 4 \rightarrow x^2 + y^2 - 4 = 0$

$g(x,y) = x^2 + y^2 - 4 = 0$

restricted domain of f are (x,y) that lie on the circle

$$\left.\begin{aligned} \nabla f &= (4x+1)\vec{i} + 2y\vec{j} \\ \lambda \nabla g &= 2\lambda x\,\vec{i} + 2\lambda y\,\vec{j} \end{aligned}\right\} \nabla f = \lambda \nabla g$$

...ical points

$\lambda \nabla g = 2\lambda x\, \mathbf{i} + 2\lambda y\, \mathbf{j}$

$$\boxed{\begin{array}{ll} ① & 4x+1 = 2\lambda x \\ ② & 2y = 2\lambda y \\ ③ & x^2+y^2 = 4 \end{array}}$$

$\leftarrow$ solve to critical points

By ② $\quad 2y - 2\lambda y = 0 \overset{\text{implies}}{\Rightarrow} 2y(1-\lambda) = 0$

$\Rightarrow y = 0$ or $\lambda = 1$.

case I: $y = 0$. By ③ $x^2 = 4 \Rightarrow x = \pm 2$

$(\pm 2, 0)$ c.p.s.

case II: $\lambda = 1$

By ① $\quad 4x + 1 = 2x \Rightarrow 2x = -1 \Rightarrow -\frac{1}{2}$

By ③ $\quad \left(-\frac{1}{2}\right)^2 + y^2 = 4 \Rightarrow y^2 = 4 - \frac{1}{4} = \frac{15}{4}$

$\Rightarrow y = \pm \frac{\sqrt{15}}{2} \Rightarrow$ c.p. $\left(-\frac{1}{2}, \pm\frac{\sqrt{15}}{2}\right)$.

x	y	$f(x,y) = 2x^2 + x + y^2 - 2$
-2	0	4 $\quad f(-2,0) = 8 - 2 - 2 = 4$
2	0	8 Abs max $\quad f(2,0) = 8 + 2 - 2 = 8$
$-\frac{1}{2}$	$\pm\frac{\sqrt{15}}{2}$	$\frac{7}{4}$ abs min

Note: This method also works for $w = f(x,y,z)$

w/ constraint surface $\underbrace{g(x,y,z) = 0}_{\text{surface}}$

ex Find the extrema using L.M.

$$f(x,y,z) = x^2y^2z^2 \; ; \quad \underbrace{x^2+y^2+z^2=1}_{\text{sphere}}$$

$$\underbrace{x^2+y^2+z^2-1=0}_{g(x,y,z)}$$

$$\nabla f = 2xy^2z^2\vec{i} + 2x^2yz^2\vec{j} + 2x^2y^2z\vec{k}$$

$$\lambda\nabla g = 2\lambda x\vec{i} + 2\lambda y\vec{j} + 2\lambda z\vec{k}$$

(1) $2xy^2z^2 = 2\lambda x \rightarrow \lambda = y^2z^2$

(2) $2x^2yz^2 = 2\lambda y \rightarrow \lambda = x^2z^2$

(3) $2x^2y^2z = 2\lambda z \rightarrow \lambda = x^2y^2$

(4) $x^2+y^2+z^2=1$

Note: when $x=0$ or $y=0$ or $z=0$, $f = 0$ (abs. min.), which is abs. min.

So Assume $x, y, z \neq 0$.

From (1), (2), (3), $\boxed{y^2z^2 = x^2z^2} = x^2y^2$

$\rightarrow y^2 = x^2 = z^2$

From (4) $x^2+x^2+x^2=1 \Rightarrow 3x^2=1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$

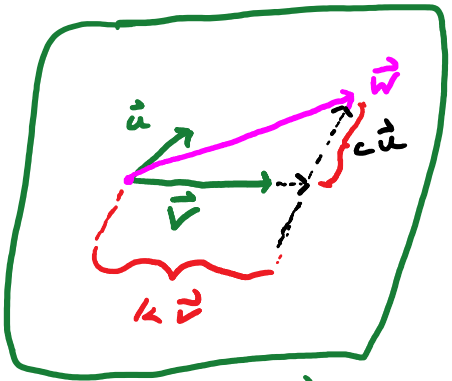
$\left(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

c.p.'s

$f(\text{Any c.p.}) = \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{1}{27}$ abs. max

Note:



$\vec{w} = c\vec{u} + k\vec{v}$

Note:
If $\vec{u}$, $\vec{v}$, and $\vec{w}$ lie in the same plane, then $\vec{w}$ can be written in terms of $\vec{u}$ and $\vec{v}$ using scalar multiplication and the triangle law.

Two Constraints

Suppose $w = f(x,y,z)$ has two constraints, $g(x,y,z) = c$ and $h(x,y,z) = k$. The domain of f is on the curve of intersection of h and g. If $f(x_0,y_0,z_0)$ is an extremum, then it turns out $\nabla f(x_0,y_0,z_0)$ is in the plane created by $\nabla g(x_0,y_0,z_0)$ and $\nabla h(x_0,y_0,z_0)$.

So, $\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0)$

Lagrange Multipliers

ex) Find the extrema using L.M.

$f(x,y,z) = 3x - y - 3z$; $\underbrace{x+y-z=0}_{g}$, $\underbrace{x^2+2z^2=1}_{h}$

$$\boxed{\nabla f = \lambda \nabla g + \mu \nabla h}$$

$\nabla f = 3\vec{i} - 1\vec{j} - 3\vec{k}$

$\lambda \nabla g = \lambda\vec{i} + \lambda\vec{j} - \lambda\vec{k}$

$\lambda \nabla g = \lambda \vec{i} + \lambda \vec{j} - \lambda k$

$\mu \nabla h = (2\mu x)\vec{i} \quad + 4\mu z \vec{k}$

$3 = \lambda + 2\mu x \longrightarrow$

$-1 = \lambda$

$-3 = -\lambda + 4\mu z \longrightarrow$

$x + y - z = 0$

$x^2 + 2z^2 = 1$

$$\boxed{\begin{array}{ll} 3 = -1 + 2\mu x & (1) \\ -3 = 1 + 4\mu z & (2) \\ x + y - z = 0 & (3) \\ x^2 + 2z^2 = 1 & (4) \end{array}}$$

From (1) and (2), $x = \dfrac{4}{2\mu} \Rightarrow x = \dfrac{2}{\mu}$

and $z = \dfrac{-4}{4\mu} \Rightarrow z = \dfrac{-1}{\mu} \leftarrow$

From (4) $\left(\dfrac{2}{\mu}\right)^2 + 2\left(\dfrac{-1}{\mu}\right)^2 = 1 \Rightarrow \dfrac{4}{\mu^2} + \dfrac{2}{\mu^2} = 1$

$\Rightarrow \dfrac{6}{\mu^2} = 1 \Rightarrow \mu^2 = 6 \Rightarrow \mu = \pm\sqrt{6}$

$x = \pm\dfrac{2}{\sqrt{6}}, \quad z = \mp\dfrac{1}{\sqrt{6}}$

(3) $x + y - z = 0$

$x = \dfrac{2}{\sqrt{6}}$: $\dfrac{2}{\sqrt{6}} + y + \dfrac{1}{\sqrt{6}} = 0 \Rightarrow y = \dfrac{-3}{\sqrt{6}}$

$x = -\dfrac{2}{\sqrt{6}}$: $-\dfrac{2}{\sqrt{6}} + y - \dfrac{1}{\sqrt{6}} = 0 \Rightarrow y = \dfrac{3}{\sqrt{6}}$

$x = -\frac{2}{\sqrt{6}}$ | $-\frac{2}{\sqrt{6}} + y - \frac{1}{\sqrt{6}} = 0 \Rightarrow y = \frac{3}{\sqrt{6}}$

$\underbrace{\left(\frac{2}{\sqrt{6}}, \frac{-3}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)}_{\vec{a}}$ $\underbrace{\left(\frac{-2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}_{\vec{b}}$ c.p.

abs. max abs. min.

$f(\vec{a}) = 2\sqrt{6}$, $f(\vec{b}) = -2\sqrt{6}$

ex Find the points on the surface $\boxed{y^2 = 9 + xz}$ that are closest to the origin.

set up only

minimizing distance squared produces same c.p.'s

constraint

$\underbrace{y^2 - xz - 9}_{g(x,y,z)} = 0$

$d^2 = \boxed{f(x,y,z) = x^2 + y^2 + z^2}$

$\nabla f = 2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}$

$\lambda \nabla g = -\lambda z\,\vec{i} + 2\lambda y\,\vec{j} - \lambda x\,\vec{k}$

$$\boxed{\begin{aligned} 2x &= -\lambda z \\ 2y &= 2\lambda y \\ 2z &= -\lambda x \\ y^2 &= 9 + xz \end{aligned}}$$

$\longrightarrow (0, \pm 3, 0)$ c.p.

$y^2 = 9 + xz$

as $x, z \to \infty$, $y \to \infty$, so There is no farthest point away on the surface (from origin). So $(0, \pm 3, 0)$

is the closest point on the surface
$y^2 = 9 + xz$ to the origin.