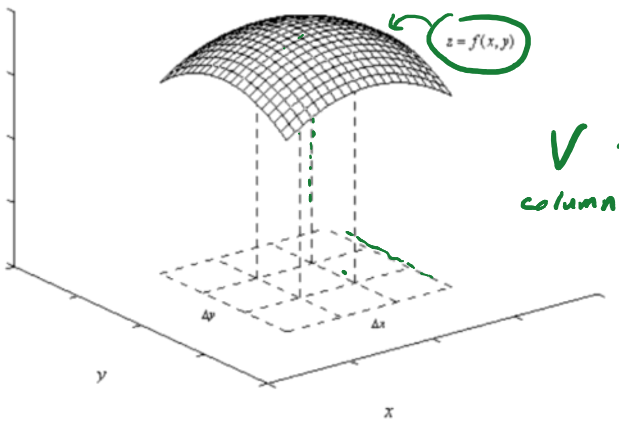
Section 15.1: Double Integrals and Volume

Thursday, March 12, 2015 12:47 PM

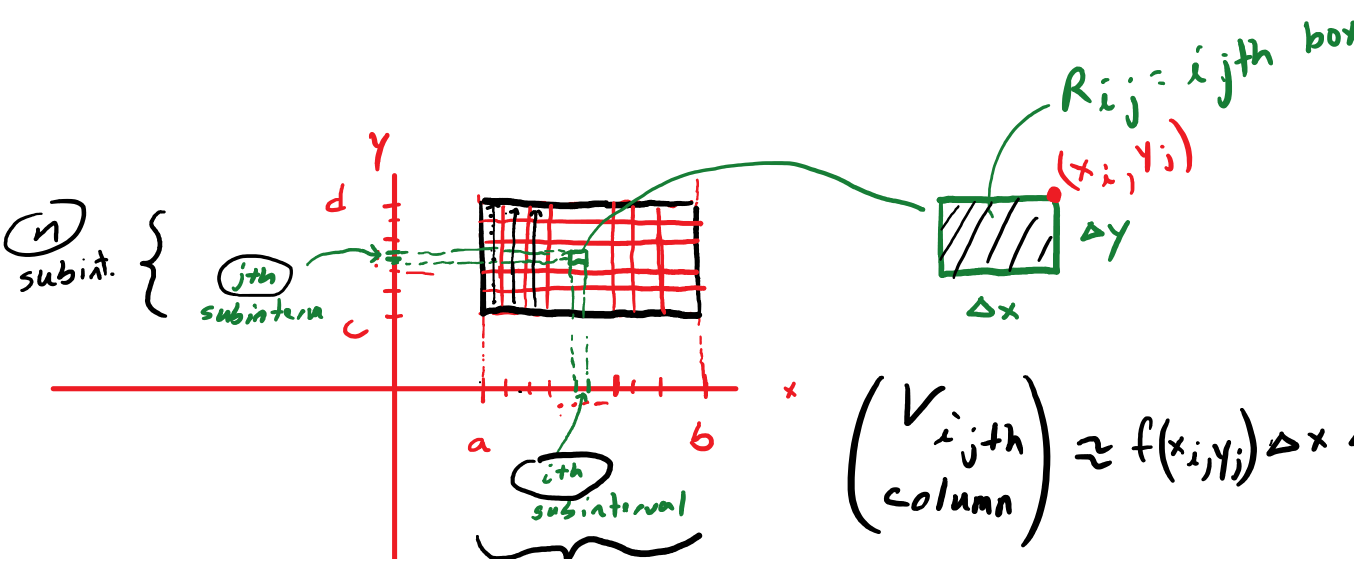
Goal: To represent the Volume of a solid using a double integral

Look at the picture below. Suppose we wish to find the volume between a rectangular region R in the xy-plane and the surface above it. We can approximate the volume by breaking up R into a grid (or mesh) consisting of smaller rectangles and then summing the volume of the columns associated with the smaller rectangles under the surface.



$$V_{\text{column}} \approx \underbrace{f(x,y)}_{\text{Ht. column}} \Delta x \Delta y$$

Let $z = f(x,y) \geq 0$. Suppose we want to find the volume between a rectangular region R in xy-plane and f.



$$\left(\begin{array}{c} V_{ij\text{th}} \\ \text{column} \end{array}\right) \approx f(x_i, y_j)\,\Delta x\,\Delta y$$

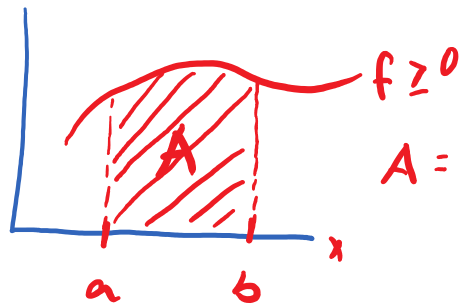
ith subinterval

$\underbrace{}_{m \text{ subinterval}}$

(column)

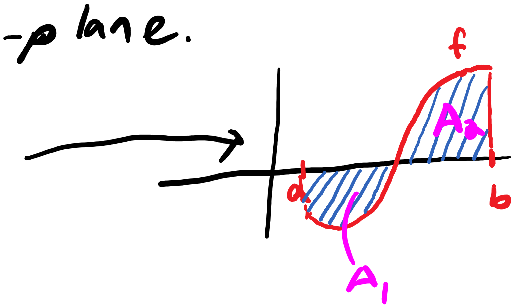
$$\begin{pmatrix}\text{Total}\\ \text{volume}\end{pmatrix} = V_{i,j} \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\,\Delta x\,\Delta y$$

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\underbrace{\Delta x\,\Delta y}_{\Delta A} = \underbrace{\iint_R f(x,y)\,dA}_{\text{double integral of } f \text{ over } R}$$



$$A = \int_a^b f(x)\,dx$$

Note: ① In general $\iint_R f(x,y)\,dA$ can be calculated by subtracting the volume below the xy-plane from volume above xy-plane.



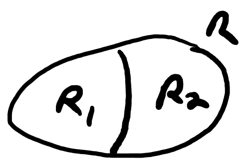
$$\int_a^b f(x)\,dx = A_2 - A_1$$

② $\iint_R f(x,y)\,dA$ exists if f is continuous.

③ a) $\iint\limits_R (f+g)\,dA = \iint\limits_R f\,dA + \iint\limits_R g\,dA$

b) $\iint\limits_R c f\,dA = c\iint\limits_R f\,dA$

c) $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \phi$



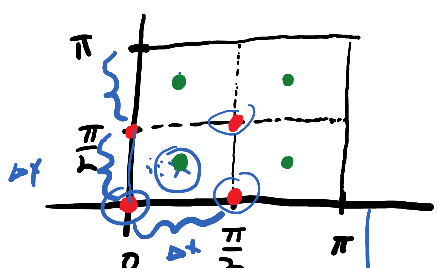
$$\iint\limits_{R=R_1\cup R_2} f\,dA = \iint\limits_{R_1} f\,dA + \iint\limits_{R_2} f\,dA$$

ex Use a Riemann sum w/ $m = n = 2$

to estimate the volume of $\iint\limits_R \underbrace{\sin(x+y)}_{f(x,y)}\,dA$

where $R = [0,\pi]\times[0,\pi]$.

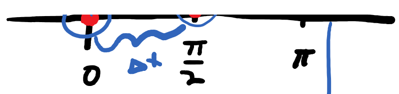
a) use lower left corners as sample points



sample pt: $(0,0)$ $(0,\frac{\pi}{2})$ $(\frac{\pi}{2},0)$ $(\frac{\pi}{2},\frac{\pi}{2})$

$\Delta x = \frac{\pi}{2}$, $\Delta y = \frac{\pi}{2}$

$\Delta A = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}$



$(\frac{\pi}{2}, \frac{\pi}{2})$

$\Delta x = \frac{\pi}{2}$, $\Delta y = \frac{\pi}{2}$

$\Delta A = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$

$f(x,y) = \sin(x+y)$

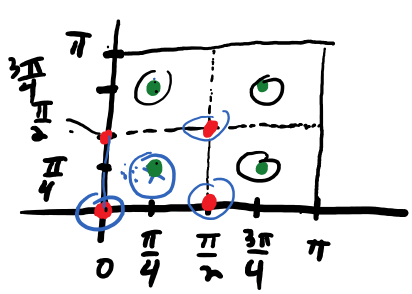
$$\iint_R f(x,y)\,dA \approx \frac{\pi^2}{4}\left(f(0,0) + f(0,\tfrac{\pi}{2}) + f(\tfrac{\pi}{2},0) + f(\tfrac{\pi}{2},\tfrac{\pi}{2}) \right)$$

$$= \frac{\pi^2}{4}\left(\sin(0+0) + \sin(0+\tfrac{\pi}{2}) + \sin(\tfrac{\pi}{2}+0) + \sin(\underbrace{\tfrac{\pi}{2}+\tfrac{\pi}{2}}_{\pi}) \right)$$

$$= \frac{\pi^2}{4}\left(0 + 1 + 1 + 0 \right)$$

$$= \frac{\pi^2}{2}$$

b) Use midpoints as sample pts.



pts: $(\frac{\pi}{4}, \frac{\pi}{4})$ $(\frac{\pi}{4}, \frac{3\pi}{4})$, $(\frac{3\pi}{4}, \frac{\pi}{4})$ $(\frac{3\pi}{4}, \frac{3\pi}{4})$

$$\iint_R \sin(x+y)\,dA = \frac{\pi^2}{4}\left[\sin(\underbrace{\tfrac{\pi}{4}+\tfrac{\pi}{4}}_{\pi/2}) + \sin(\underbrace{\tfrac{\pi}{4}+\tfrac{3\pi}{4}}_{\pi}) + \sin(\tfrac{3\pi}{4}+\tfrac{\pi}{4}) + \sin(\underbrace{\tfrac{3\pi}{4}+\tfrac{3\pi}{4}}_{3\pi/2}) \right]$$

$$= \frac{\pi^2}{} \left[1 + 0 + 0 - 1 \right]$$

R

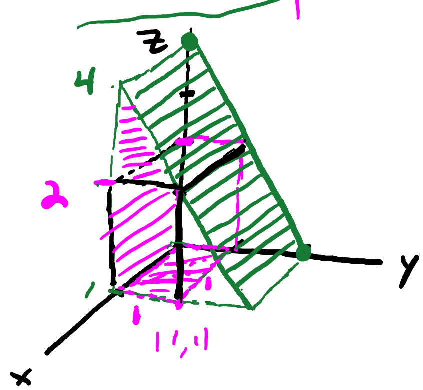
$$= \frac{\pi^2}{4}\left[1 + 0 + 0 - 1\right]$$

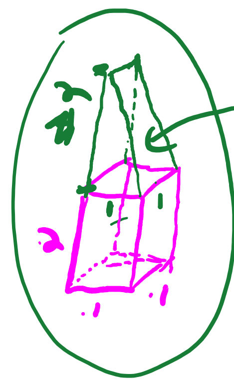
$$= 0$$

ex Evaluate using geometry

$$\iint_R \underbrace{(4-2y)}_{z}\, dA, \quad R = [0,1]\times[0,1]$$

$$z = 4 - 2y$$





Volume is $\iint_R (4-2y)\,dA$

$$\iint_R (4-2y)\,dA = V_{Box} + V_{wedge}$$

$$= 1\cdot 1\cdot 2 + \tfrac{1}{2}\cdot 1\cdot 1\cdot 2$$

$$= 2 + 1$$

$$= 3 \text{ units}^3$$

Def: The Average value of $z = \boxed{f(x,y)}$ on R.

★ $$\boxed{f_{ave} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA}$$

#5 in HW.