Section 16.3: FTLI

Tuesday, April 21, 2015 3:22 PM

Goal: To evaluate $\int_C \nabla f \cdot d\vec{r}$

$f(x,y,z)$ a potential fctn.

Assumptions: $\vec{F} = \nabla f = \langle f_x, f_y, f_z \rangle$ is continuous on a simply connected open region R, and C is piecewise smooth in R.

no holes or separations

Note (FTLI): $\boxed{\int_C \vec{F}\cdot d\vec{r}} = \int_C \nabla f \cdot d\vec{r} = \int_C \langle f_x, f_y, f_z \rangle \cdot \langle dx, dy, dz \rangle$

$$= \int_C \underbrace{f_x\,dx + f_y\,dy + f_z\,dz}_{df}$$

$$= \int_C df$$

$$= \int_a^b \frac{df}{dt}\,dt$$

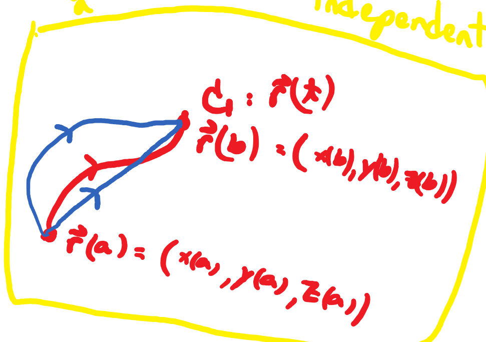
$$= \Big[f\big(x(t), y(t), z(t)\big)\Big]_{t=a}^{t=b} \quad \text{(FTC)}$$

$$= f\big(x(b), y(b), z(b)\big) - f\big(x(a), y(a), z(a)\big)$$

$$= \boxed{f(\text{end pt.}) - f(\text{beginning})}$$

FTLI

$\int_a^b \nabla f \cdot d\vec{r}$ is path independent



ex Let $\vec{F} = \nabla f$ where $f(x,y,z) = -\dfrac{1}{x^2+y^2+z^2}$.

Find the work done in moving a particle along smooth C joining $(1,0,0)$ to $(0,0,2)$.

$$W = \int_C \vec{F}\cdot d\vec{r} \overset{\text{FTLI}}{=} f(0,0,2) - f(1,0,0)$$

$$= -\frac{1}{4} - \left(-\frac{1}{1+0^2+0^2}\right)$$

$$= -\frac{1}{4} + 1$$

$$= \frac{3}{4}$$

Theorem: Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$. $\vec{F}$ is conservative $\boxed{\text{iff}}$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Pf: $(\Rightarrow)$ Assume $\vec{F}$ is conservative, so $\vec{F} = \langle \overset{P}{\overset{"}{f_x}}, \overset{Q}{\overset{"}{f_y}} \rangle$ where $f(x,y)$ is a potential fctn. Now $\underbrace{f_{xy}}_{\frac{\partial P}{\partial y}} = \underbrace{f_{yx}}_{\frac{\partial Q}{\partial x}}$ (mixed partials are equal)

$(\Leftarrow)$ Assume $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Proof coming in next section.

ex Show $\vec{F}(x,y) = \overbrace{ye^{xy}}^{P = f_x}\,\vec{i} + \overbrace{xe^{xy}}^{Q = f_y}\,\vec{j}$ is conservative and find its potential function

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \vec{F} \text{ conservative}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \vec{F} \text{ conservative}$$

$$\frac{\partial P}{\partial y} = 1 \cdot e^{xy} + xy\, e^{xy}$$

$$\frac{\partial Q}{\partial x} = 1 \cdot e^{xy} + xy\, e^{xy}$$

equal

so $\vec{F}$ is conservative.

$$f(x,y) = \int f_x(x,y)\,dx = \int y\, e^{xy}\,dx$$

$$f(x,y) = \frac{\cancel{y}}{\cancel{y}}\, e^{xy} + g(y)$$

$$f_y(x,y) = \boxed{x e^{xy} + \underbrace{g'(y)} = x e^{xy} + 0}$$

$$g'(y) = 0$$

$$g(y) = c, \quad c \text{ a constant}$$

$$f(x,y) = e^{xy}$$

ex Find a potential function for $\vec{F}(x,y,z) = \overbrace{yz}^{f_x}\,\vec{i} + \overbrace{xz}^{f_y}\,\vec{j} + \overbrace{(xy + 2z)}^{f_z}\,\vec{k}$

$$f(x,y,z) = \int f_x\,dx = \int yz\,dx = xyz + \underbrace{g(y,z)}_{\text{a constant w.r.t. } x}$$

$$f_y(x,y,z) = xz + g_y(y,z) = xz + 0$$

$$g_y(y,z) = 0 \to g(y,z) = h(z)$$

so $g_y(y,z) = 0 \to g(y,z) = \underbrace{h(z)}_{\text{a constant w.r.t } y}$

so $f(x,y,z) = xyz + h(z)$

$f_z(x,y,z) = xy + h'(z) = xy + 2z$

so $h'(z) = 2z \to h(z) = z^2 + c$, c a constant

Thus, $\boxed{f(x,y,z) = xyz + z^2}$

A potential function of $\vec{F}$
(taking $c=0$)