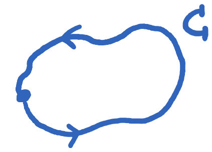
Section 16.4: Green's Theorem

Monday, April 27, 2015 7:51 PM

Goal: To use Green's Theorem to evaluate line integrals



Recall: A curve C in a plane is **positively oriented** if it is traveled in a counterclockwise direction.

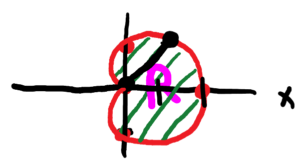
Theorem 1: Green's Theorem

Let C be a positively oriented, piecewise smooth, simple closed curve that bounds a region R in the xy-plane. Then, as long as P and Q have continuous partial derivatives in an open region containing R,

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Idea of Proof: Show $\int_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\,dA$ and $\int_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\,dA$, then sum to get the result.

ex Evaluate $\int_C \overbrace{(x^2-y^2)}^{P}\,dx + \overbrace{2xy}^{Q}\,dy$, C: $\underbrace{r = 1+\cos\theta}_{\text{polar}}$



$$\int_C (x^2-y^2)\,dx + 2xy\,dy = \boxed{\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

$$= \iint_R (2y - (-2y))\,dA$$

$$= \iint_R 4y\,dA$$

$$= 4\int_0^{2\pi}\int_0^{1+\cos\theta} r\sin\theta\, r\,dr\,d\theta$$

2π $1+\cos\theta$

$$= 4\int_0^{2\pi}\int_0^{1+\cos\theta} r^2\sin\theta\,dr\,d\theta$$

$$= \frac{4}{3}\int_0^{2\pi}\sin\theta\,\left[r^3\right]_0^{1+\cos\theta}\,d\theta$$

$$= -\frac{4}{3}\int_0^{2\pi}\overbrace{-\sin\theta\,(1+\cos\theta)^3\,d\theta}^{u^3\,du}$$

$u = 1+\cos\theta$

$du = -\sin\theta\,d\theta$

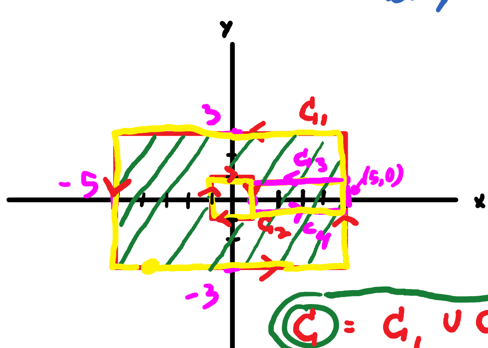
$$= -\frac{\cancel{4}}{3}\cdot\frac{1}{\cancel{4}}\left[(1+\cos\theta)^4\right]_0^{2\pi}$$

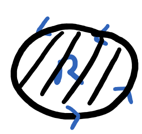
$$= -\frac{1}{3}\left[(1+1)^4-(1+1)^4\right]$$

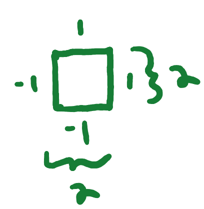
$$= 0$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

ex Evaluate $\displaystyle\int_C (\overbrace{y-x}^{P})\,dx + (\overbrace{2x-y}^{Q})\,dy$ where C is the boundary of . . . .







$C = C_1 \cup C_2$

$C = C_1 \cup C_2 \cup C_3 \cup C_4$

$$\int_C = \int_{C_1\cup C_2\cup C_3\cup C_4} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$\int_{C_1 \cup C_2 \cup C_3 \cup C_4} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$= \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{-C_3}$$

$$\int_{C_1 \cup C_2} = \int_{C_1} + \int_{C_2} + \cancel{\int_{C_3}} - \cancel{\int_{C_3}}$$

$$\int_C F \cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_R (2-1)\, dA$$

$$= \iint_R 1\, dA = \text{Area of } R = 10 \cdot 6 - 2 \cdot 2$$

$$= 56$$

Theorem 2: Corollary to Green's Theorem

If R is a plane region bounded by a piecewise smooth simple closed curve C, oriented counterclockwise, then the area of R is given by

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

$$\frac{\partial Q}{\partial x} = 1, \quad \frac{\partial P}{\partial y} = -1 \qquad A = \boxed{\frac{1}{2}\int_C x\,dy - y\,dx} = \boxed{\int_C x\,dy} = -\int_C y\,dx \qquad P, Q = 0$$

$$\frac{1}{2}(1-(-1)) = \frac{1}{2}(2) = 1$$

$$\int_C P\,dx + Q\,dy = \iint_R \boxed{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)} dA = \text{Area of } 1$$

ex Use a line integral to find the area ~~of~~ enclosed by a circle of radius a.

$$A = \boxed{\int_C x\,dy}$$

$$C: \quad x^2 + y^2 = a^2 \longrightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$$

$$\boxed{x = a\cos t, \quad y = a\sin t}, \quad 0 \le t \le 2\pi$$

$$dy = a\cos t\,dt$$

$$A = \int_0^{2\pi} a\cos t \, a\cos t \, dt$$

$$= a^2 \int_0^{2\pi} \cos^2 t \, dt$$

$$= \frac{a^2}{2} \int_0^{2\pi} (1 + \cos 2t)\, dt$$

$$= \frac{a^2}{2} \left[t + \frac{1}{2}\sin 2t \right]_0^{2\pi}$$

$$= \frac{a^2}{2} \left[(2\pi + 0) - 0 \right]$$

$$= \pi a^2$$

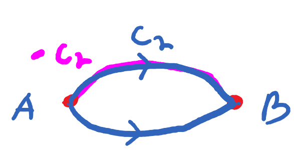
Theorem 3: Let $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ have continuous first partial derivatives in an open connected region R, and let C be a piecewise smooth curve in R. The following are equivalent

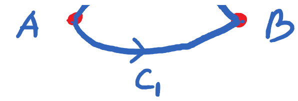
FTLI

a) $\mathbf{F}$ is conservative. $\vec{F} = \nabla f$

b) $\int_C \mathbf{F} \bullet d\mathbf{r}$ is independent of path.

c) $\int_C \mathbf{F} \bullet d\mathbf{r} = 0$ for every closed curve C in R.





$$\int_{C_1 \cup (-C_2)} \vec{F} \cdot d\vec{r} = 0$$

$$= \int_{C_1} + \int_{-C_2} = 0$$

$$= \int_{C_1} - \int_{C_2} = 0$$

$$\boxed{\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}}$$

Thus, the integral is path independent

Note: If $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$ then for every simple closed curve, C_1, enclosing R...

$$\boxed{\int_{C_1} \vec{F} \cdot d\vec{r}} = \int_{C_1} P\,dx + Q\,dy = \iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{0} dA = \boxed{0},$$

any C_1

which means $\vec{F}$ is conservative.

by the above Theorem.

So if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, then $\vec{F}$ is conservative