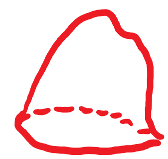
Section 16.7: Surface Integrals

Wednesday, April 29, 2015 8:48 PM

Note: In this section S is described by either

$z = g(x,y)$ or $\vec{r}(u,v)$.



$S: z = g(x,y)$

Def: The Surface Integral of f over S

$$\iint_S f(x,y,z)\, dS = \iint_R f(x,y,z)\sqrt{1+g_x^2+g_y^2}\, dA$$

where $z = g(x,y)$

Note: If $f = 1$, then integral gives surface area of S on the domain.

ex) Evaluate $\iint_S \overbrace{(x-2y+z)}^{f}\, dS$, $\overbrace{z = \underbrace{15-2x+3y}_{g}}^{S:\ g(x,y)=z}$, $0 \le x \le 2$, $0 \le y \le 4$

$$\iint_S f\, dS = \iint_R f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}\, dA$$

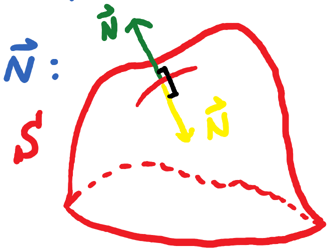
$$g_x = -2, \quad g_y = 3$$

$$\iint_R (x-2y+15-2x+3y)\sqrt{1+4+9}\, dA$$

$$\int_0^2 \int_0^4 (-x+y+15)\sqrt{14}\, dy\, dx = 128\sqrt{14}$$

Surface Orientation

Two ways to orient a surface using a unit normal vector $\vec{N}$:



For upward $\vec{N}$, $\vec{k}$ component is positive

For downward $\vec{N}$, $\vec{k}$ component is negative

Orient S by choosing an orientation for $\vec{N}$ and stick with it. Now, if S is given by $z = g(x,y)$, then define

$z - g(x,y) = 0$

$\boxed{G(x,y,z) = z - g(x,y)}$ and recall ∇G is normal to S (since S is a level surface of $G(x,y,z)$). Then

unit normal

$$\vec{N} = \frac{\nabla G}{\|\nabla G\|} = \boxed{\frac{-g_x\vec{i} - g_y\vec{j} + \vec{k}}{\sqrt{1+g_x^2+g_y^2}}} \quad \text{or} \quad \vec{N} = \frac{-\nabla G}{\|\nabla G\|}$$

up

downward

Notes: (1) For closed S, $\vec{N}$ always points outward.

(2) $\vec{N}$ must vary continuously for S to be orientable

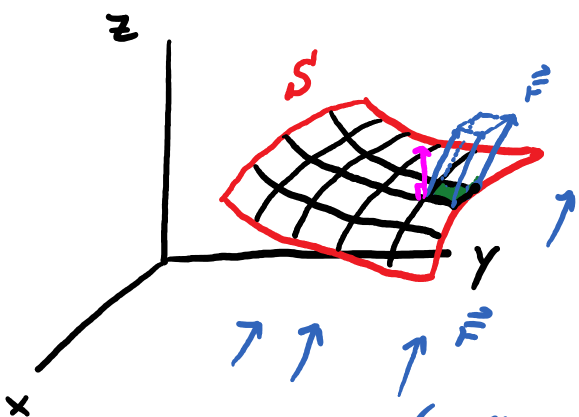
(3) If S is defined by $\vec{r}(u,v)$, then

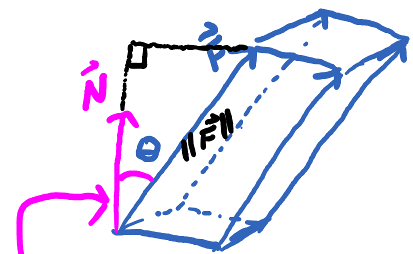
$$\boxed{\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}} \quad \left(\text{or perhaps } \frac{-(\vec{r}_u \times \vec{r}_v)}{\|\vec{r}_u \times \vec{r}_v\|}\right)$$

$$\boxed{\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}} \qquad \left(\text{or perhaps } \frac{-(\vec{r}_u \times \vec{r}_v)}{\|\vec{r}_u \times \vec{r}_v\|}\right)$$

Flux Integrals

Flux is the quantity of "fluid" passing through a surface per unit of time.





$$\left(\begin{array}{c}\text{Flux through}\\ \text{patch of surface}\end{array}\right) \approx \left(\begin{array}{c}\text{volume}\\ \text{of "parallepiped"}\end{array}\right)$$

$$= (\text{height})(\text{Area Base})$$

$$= \|\vec{F}\| \cos\theta \cdot dS$$

$$= \underbrace{\|\vec{F}\|\|\vec{N}\| \cos\theta}_{\vec{F}\cdot\vec{N}} \cdot dS$$

$$= \underbrace{\vec{F}\cdot\vec{N}\, dS}_{\text{Flux Element}}$$

Def: The flux integral of $\vec{F}$ across S is ...

$$\boxed{\iint_S \vec{F}\cdot\vec{N}\,dS}$$

where $\vec{F} = \langle P, Q, R \rangle$ has continuous first partials on smooth S

<u>Notes</u>: ① $\underbrace{\boxed{\iint_S \vec{F}\cdot\vec{N}\,dS}}_{\text{flux}} = \boxed{\iint_R \vec{F}\cdot(\pm\nabla G)\,dA}$ ← calculation formula

$$\left[\text{since } \vec{F}\cdot\vec{N}\,dS = \vec{F}\cdot\left(\pm\frac{\nabla G}{\cancel{\|\nabla G\|}}\right)\cancel{\sqrt{1+g_x^2+g_y^2}}\,dA\right]$$

② If $\rho(x,y,z)$ is the density of the fluid at (x,y,z) then $\iint_S \rho\,\vec{F}\cdot\vec{N}\,dS$ gives mass of fluid through S per unit of time

③ If S is given by $\vec{r}(u,v)$, then

$$\iint_S \vec{F}\cdot\vec{N}\,dS = \iint_D \vec{F}\cdot(\pm\,\vec{r}_u\times\vec{r}_v)\,dA$$

ex Consider $\vec{F}(x,y,z) = (x+y)\,\vec{i} + y\,\vec{j} + z\,\vec{k}$; S is the closed surface given by $\boxed{z = 1-x^2-y^2}$, and $z=0$. Find $\iint_S \vec{F}\cdot\vec{N}\,dS$.

$\leftarrow g$ gives S_1

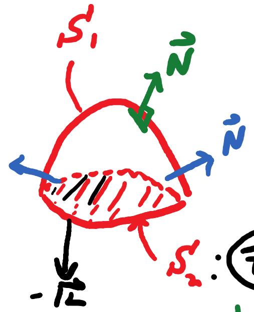
$0 = 1-x^2-y^2$

$x^2+y^2 \le 1$

S_1 $\vec{N}$

S $0 = 1 - x^2 - y^2$

$x^2 + y^2 \le 1$



$S = S_1 \cup S_2$

S_2 : $z = 0$ (which is also upward on S_1)

outward

$$\iint_{S_1} \vec{F} \cdot \vec{N}\, dS = \iint_{S_1} \vec{F} \cdot \nabla G \, dA$$

$$G_1(x,y,z) = x^2 + y^2 + z - 1$$

$$\boxed{\nabla G_1 = 2x\,\vec{i} + 2y\,\vec{j} + 1\vec{k}}$$

$$\boxed{\vec{F}(x,y,z) = (x+y)\,\vec{i} + y\,\vec{j} + z\,\vec{k}}$$

$$\vec{F} \cdot \nabla G_1 = 2x^2 + 2xy + 2y^2 + z$$

$g(x,y) = 1 - x^2 - y^2$

$$= x^2 + y^2 + 2xy + 1$$

$$\iint_{S_1} \vec{F} \cdot \vec{N}\, dS = \iint_{R:\ x^2+y^2 \le 1} (\underbrace{x^2 + y^2}_{r^2} + 2xy + 1)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (r^2 + 2r^2 \cos\theta \sin\theta + 1)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (r^3 + 2r^3 \cos\theta \sin\theta + r)\, dr\, d\theta$$

$$= \frac{3\pi}{2}$$

$$\iint_{S_2} \vec{F}\cdot\vec{N}\,dS = \iint_{S_2} \vec{F}\cdot(-\vec{k})\,dS$$

$$\iint_{S_2} z\,dS$$

$$\uparrow$$

$$\iint_{S_2} 0\,dS$$

$$\iint_{S = S_1 \cup S_2} \vec{F}\cdot N\,dS = \iint_{S_1} \vec{F}\cdot\vec{N}\,dS + \iint_{S_2} \vec{F}\cdot\vec{N}\,dS$$

$$= \frac{3\pi}{2} + 0$$

$$= \frac{3\pi}{2}$$