Section 16.8: Stokes' Theorem

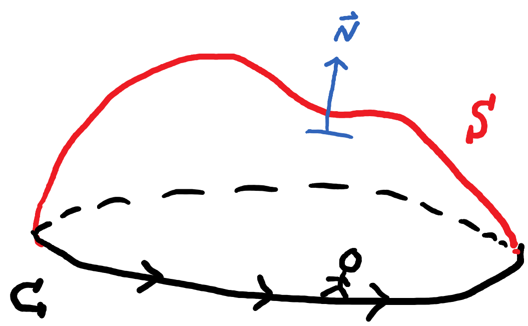
Monday, May 4, 2015 8:08 PM

Goal: To evaluate a line integral over a vector field using Stokes' Theorem.

Stokes':
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \underbrace{(\text{curl}\,\vec{F})\cdot\vec{N}\,dS}_{\text{circulation element}}$$

Where S is an oriented piecewise smooth surface bounded by C, a piecewise smooth, positively oriented closed curve.

Also, $\vec{F}$ has continuous 1st partials in an open region containing S.



Notes:

① Green's Theorem is a special case of Stokes' Theorem with S devolving into R and $\vec{N}$ into $\vec{k}$.

$$\text{curl}\,\vec{F}\cdot\vec{N} = \text{curl}\,\vec{F}\cdot\vec{k} = \left\langle\ ,\ ,\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right\rangle\cdot\langle 0,0,1\rangle$$
$$= \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$$

② For "small" C and S with a point P on S, $\vec{F}\approx\underbrace{\vec{F}(P)}_{\text{constant}}$ and $\vec{N}\approx\underbrace{\vec{N}(P)}_{\text{constant}}$. So by Stokes' Theorem...

$\vec{F} \approx \underbrace{\vec{F}(P)}_{\text{constant}}$ and $\vec{N} \approx \underbrace{\vec{N}(P)}_{\text{constant}}$. So by Stokes' Theorem...

$$\int_C \vec{F} \cdot d\vec{r} \approx \iint_S \operatorname{curl} \vec{F}(P) \cdot \vec{N}(P)\, dS$$

$$\text{"} \quad \approx \operatorname{curl} \vec{F}(P) \cdot \vec{N}(P) \iint_S dS$$

$$\underbrace{\int_C \vec{F} \cdot d\vec{r}}_{\int_C \vec{F} \cdot \vec{T}\, ds} \approx \operatorname{curl} \vec{F}(P) \cdot \vec{N}(P) \underbrace{\iint_S dS}_{\text{Area of } S}$$

$$\underbrace{\int_C \vec{F} \cdot \vec{T}\, ds}$$

measures the circulation (or flow) of $\vec{F}$ around C

(since $\vec{F} \cdot \vec{T}$ larger when $\vec{F}$ & $\vec{T}$ point in same direction: $\vec{F} \cdot \vec{T} = \|\vec{F}\| \|\vec{T}\| \cos\theta$)

③ $$\underbrace{\int_C \vec{F} \cdot d\vec{r}} = \operatorname{curl} \vec{F}(P) \cdot \vec{N}(P) \,(\text{Area of } S)$$

$$\frac{\text{circulation}}{\text{Areas of } S} = \operatorname{curl} \vec{F}(P) \cdot \vec{N}(P)$$

$$\operatorname{curl} \vec{F}(P) \cdot \vec{N}(P) = \frac{\text{circulation}}{\text{Area}} = \text{circulation per unit area}$$

$$= \text{rate of circulation}$$

④ $$\int_C \vec{F} \cdot d\vec{r} = \iint_S \overbrace{\operatorname{curl} \vec{F} \cdot \vec{N}\, dS}^{\text{circulation element}}$$

So, the curling of fluid $\vec{F}$ is biggest when $\operatorname{curl} \vec{F}$ points

④ $\underbrace{\int_C \vec{F}\cdot d\vec{r}}_{} = \underbrace{\iint_S \operatorname{curl}\vec{F}\cdot\vec{N}\,dS}_{}$ } fluid $\vec{F}$ is biggest when curl $\vec{F}$ points in the same direction as $\vec{N}$.

(circulation of $\vec{F}$ around boundary C) = (Adding up the curling of $\vec{F}$ on S)

⑤ If $\operatorname{curl}\vec{F} = \vec{0}$ throughout the domain of $\vec{F}$, then $\vec{F}$ is called irrotational (and conservative)

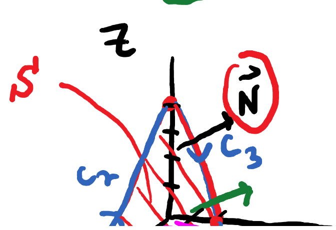
⑥ For a given boundary C of S, $\underbrace{\iint_S \operatorname{curl}\vec{F}\cdot\vec{N}\,dS}_{\int_C \vec{F}\cdot d\vec{r}}$ is independent of S.

$$\iint_{S_1} \operatorname{curl}\vec{F}\cdot\vec{N}\,dS = \iint_{S_2} \operatorname{curl}\vec{F}\cdot\vec{N}\,dS$$

Ex Let $\vec{F} = y\vec{i} + z\vec{j} - xy\vec{k}$; $S: \boxed{z = 4 - x - 2y}$ in 1st octant. Let C be the curve created when S intersects the coordinate planes. Find $\boxed{\int_C \vec{F}\cdot d\vec{r}}$.

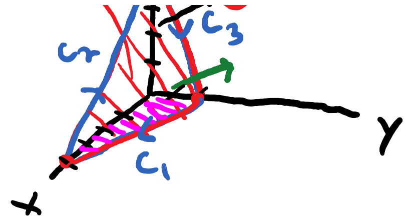
$S: \overset{g}{z} = 4 - x - 2y$

$\boxed{x + 2y + z = 4}$



$C = C_1 \cup C_2 \cup C_3$

$$\boxed{x + 2y + z = 4}$$



$$G(x,y,z) = 1z + 1x + 2y - 4$$

$$\nabla G = \vec{i} + 2\vec{j} + \vec{k}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$$

$$\overset{\text{stokes}}{=} \iint_S \operatorname{curl}\vec{F}\cdot\vec{N}\,dS = \iint_{S'} \operatorname{curl}\vec{F}\cdot(\nabla G)\,dA$$

$$\vec{F} = y\vec{i} + z\vec{j} - xy\vec{k}$$

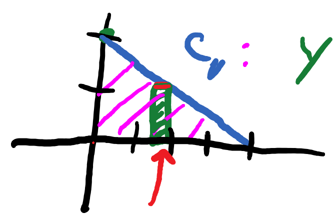
$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & -xy \end{vmatrix} = (-x-1)\vec{i} - (-y-0)\vec{j} + (0-1)\vec{k}$$

$$= (-x-1)\vec{i} + (y)\vec{j} - \vec{k}$$

$$\nabla G = (1)\vec{i} + (2)\vec{j} + \vec{k}$$

$$\operatorname{curl}\vec{F}\cdot\nabla G = -x - 1 + 2y - 1$$

$$= -x + 2y - 2$$



C_1: $y = -\frac{1}{2}x + 2$

$$\int_0^4 \int_0^{-\frac{1}{2}x+2} (-x + 2y - 2)\,dy\,dx = -8$$