Inverse and Composite Functions

Goals:

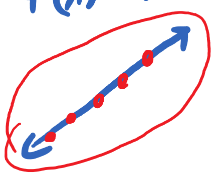
1. To identify a one-to-one function using the Horizontal Line Test.
2. To find the inverse of a given function.
3. To compose functions.

Definitions:

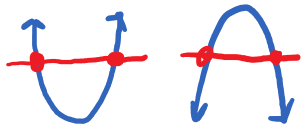
1. A function is **one-to-one** if none of its ordered pairs have the same second coordinate.

$f(x) = ax^2 + bx + c$

$f(x) = mx + b$

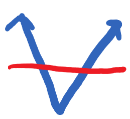


1 to 1



Not 1 to 1

$f(x) = |x|$



Not 1 to 1

1 to 1

1 to 1

2. The **inverse** of a function, f, is the set of all ordered pairs obtained by reversing the coordinates of each ordered pair that comprises f.

Not 1 to 1

ex

$f = \{(1, 2), (3, 4), (5, 6), (7, 2)\}$

D: $\{1, 3, 5, 7\}$

R: $\{2, 4, 6\}$

$f^{-1} = \{(2, 1), (4, 3), (6, 5), (2, 7)\}$

"f - inverse"

f^{-1} Not a fctn.

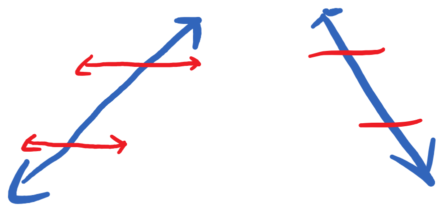
Notes:

1. Every one-to-one function has an inverse that is also a function.
2. The domain and range of a one-to-one function and its inverse

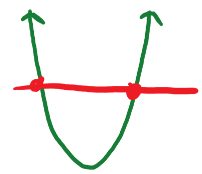
function are flip-flopped.

Horizontal Line Test: If a horizontal line crosses the graph of a function at more than one point, then the function is not one-to-one.

$y = mx + b$



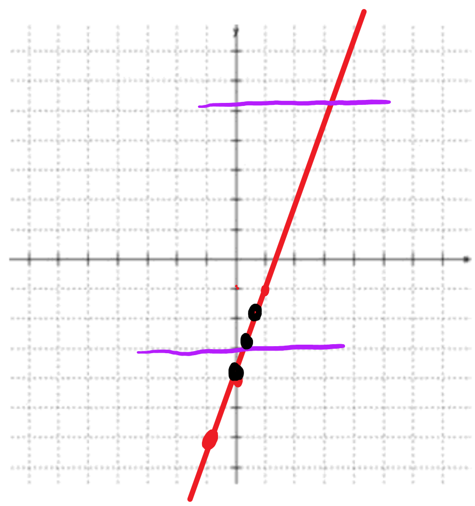
1 to 1 by HLT



Not 1 to 1 by HLT

ex) Find the inverse:

a) $\boxed{f(x) = 3x - 4}$ $m = \frac{3}{1}$



√ ① check to see if f is 1 to 1.

f is 1 to 1 by HLT.

√ ② Replace f(x) w/ y

$\boxed{y = 3x - 4}$

√ ③* Switch x and y

$x = 3y - 4$

* ④ Solve for y

$$3y - 4 = x$$
$$+4 \qquad +4$$

$$\frac{3y}{3} = \frac{x+4}{3}$$

$$\frac{3y}{3} = \frac{x+4}{3}$$

$$y = \frac{x+4}{3} = \boxed{\frac{1}{3}x + \frac{4}{3}}$$

$$y = \frac{1}{3}x + \frac{4}{3}$$

"f-inverse of x"

⑤ Replace y w/ $f^{-1}(x)$

$$f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$$

b) $g(x) = 2x^3 - 5$

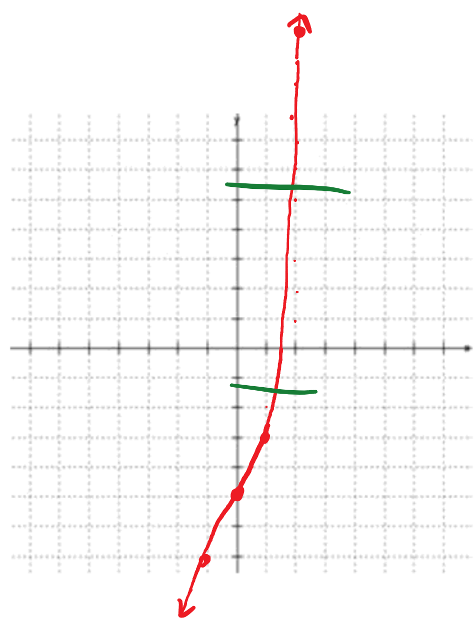
①

x	g(x)
-2	-21
-1	-7
0	-5
1	-3
2	11

$2(-2)^3 - 5 = -16 - 5$

$2(-1)^3 - 5 = -7$

g is 1 to 1 by HLT.



② $y = 2x^3 - 5$

③ $x = 2y^3 - 5$

④
$$\begin{aligned} 2y^3 - 5 &= x \\ +5 \quad &\quad +5 \\ \hline 2y^3 &= x + 5 \end{aligned}$$

$$\frac{2y^3}{2} = \frac{x+5}{2}$$

$$\sqrt[3]{y^3} = \sqrt[3]{\frac{x+5}{2}}$$

$$y = \sqrt[3]{\frac{x+5}{2}}$$

5 $\quad g^{-1}(x) = \sqrt[3]{\frac{x+5}{2}}$

$$f(x) = a(x-h)^2 + k$$

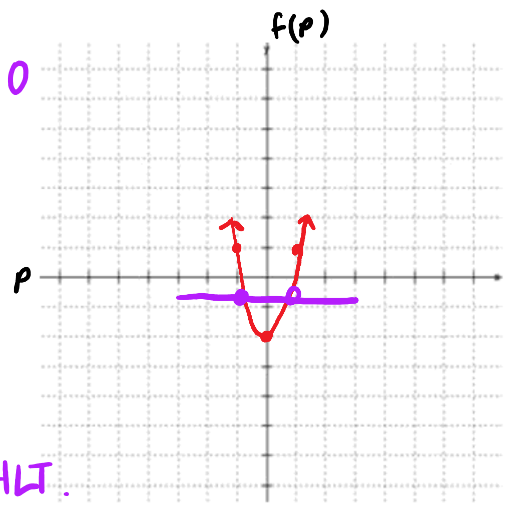
c) $f(p) = \boxed{3p^2 - 2}$, $p \geq 0$

1

p	y = f(x)
1	1
V(0	-2)
-1	1

$3(1)^2 - 2$

sym



Not 1to1 by HLT.

Composite Functions

$g \circ f$ = "g composed with f"

$$(g \circ f)(x) \overset{def}{=} g[f(x)]$$

function of a function.

function of a function.

ex) Let $f(x) = -2x+1$ and $g(x) = \boxed{x^2+3}$.

Find ...

a) $(f \circ g)(2)$ "f composed with g of 2"

$\overset{def}{=} f[\underbrace{g(2)}_{7}]$

$g(2) = (2)^2+3$
$= 7$

$= f(7)$
$= -2(7)+1$
$= -14+1$
$= -13$

compose input

$f(x) = -2x+1$ $g(x) = \boxed{x^2+3}$

b) $(f \circ g)(x)$

work from inside out

input

$\overset{def}{=} f[g(x)]$
$= f[x^2+3]$
$= -2(x^2+3)+1$
$= -2x^2-6+1$
$= -2x^2-5$

c) $(g \circ f)(x)$

$$\overset{def}{=} g[f(x)]$$

input for g

$$= g[\underbrace{-2x+1}]$$

$$= (-2x+1)^2 + 3$$

$$= 4x^2 - 4x + 1 + 3$$

$$= \boxed{4x^2 - 4x + 4}$$

Note: For inverse functions, f and f^{-1}, ...

Big Property true iff we're talking inverse function

$$(f \circ f^{-1})(x) = \boxed{f[f^{-1}(x)] = x = f^{-1}[f(x)]} = (f^{-1} \circ f)(x)$$

$f(x) = \sqrt{x}$

$g(x) = x^2$

find $x > 0$.

$$x^2 = 9$$

$$\sqrt{x^2} = \sqrt{9}$$

$$x = 3$$

(ex) use function composition to show that $f(x) = \boxed{\dfrac{x+6}{3}}$ and $g(x) = \boxed{3x-6}$ are inverses.

$$f[g(x)]$$

$$= f[3x-6]$$

$$= \frac{(3x-6)+6}{3}$$

$$= \frac{3x}{3}$$

$$g[f(x)]$$

$$= g\left[\frac{x+6}{3}\right]$$

$$= \cancel{3}\left(\frac{x+6}{\cancel{3}}\right) - 6$$

$$= x + 6 - 6$$

3

$= \frac{3x}{3}$

$= x$ ✓

$= x+6 \quad -6$

$= x$ ✓

f and g are inverses.

(ex) Let $H(x) = \sqrt{\underbrace{2x+1}_{g(x)}}$. Find f and g such that $f[g(x)] = H(x)$.

grouping symbol

let $\boxed{\begin{array}{l} g(x) = 2x+1 \\ f(x) = \sqrt{x} \end{array}}$

"inner function"

"outer function"

answer

$f[g(x)] = f(\overset{\text{input}}{2x+1}) = \sqrt{2x+1}$

(ex) The function $\boxed{I(f) = 12f}$ gives the number of inches in f feet. The function $\boxed{f(y) = 3y}$ gives the number of feet in y yards. What does $I[f(y)]$ represent?

inches

output of inches

input y yards

The number of inches in y yards.

(ex) convert 10 yards to inches.

$I[f(y)]$

$= I[\underbrace{f(10)}]$

$= I[\underbrace{f(10)}]$

$= I[3 \cdot 10]$

$= I[\overset{f}{30}]$

$= 12 \cdot 30$

$= 360$ inches

CW 9.1

① $f(x) = 4x^2 - 3 \qquad g(x) = 2x - 7$

find $(f \circ g)(x) = f[g(x)]$

$= f[\underbrace{2x-7}]$

$= 4(2x-7)^2 - 3$

$= 4(4x^2 - 28x + 49) - 3$

$= 16x^2 - 112x + 196 - 3$

$= 16x^2 - 112x + 193$

② Let $\boxed{H(x) = 2(\overset{f(x)}{x-3})^5}$. Find $f(x)$ and $g(x)$ such that $H(x) = g[f(x)]$

$f(x) = x - 3$

$g(x) = 2x^5$