Exponential and Logarithmic Equations

Goal: Let's solve these things!

$\underline{Recall:}$ ① $a^y = a^x$ iff $x = y$ $(a > 0, a \neq 1)$

② $\log_a(a^x) = x \xrightarrow{\text{exp:}} a^x = a^x$

③ $\log_a x = \log_a y$ iff $x = y$

$f(f^{-1}(x)) = x$

(ex) solve the exponential equation

a) $4^{3x} = 32$

$(2^2)^{3x} = 2^5$

$2^{6x} = 2^5$

$6x = 5$

$x = \frac{5}{6}$

b) $2 \cdot 3^x = 8$

$\dfrac{2 \cdot 3^x}{2} = \dfrac{8}{2}$

$3^x = 4$

switch to log form

$\log(3^x) = \log(4)$

$(\text{power rule}) \rightarrow \dfrac{x \cdot \log(3)}{\log 3} = \dfrac{\log(4)}{\log 3}$

$x = \dfrac{\log 4}{\log 3}$

$x = \log_3(4)$

or

$x = \dfrac{\log(4)}{\log(3)}$

c) $9^{7-3x} = 5$

power rule

$\log(9^{7-3x}) = \log(5)$

d) $e^{3x} = 10$ $\ln e = 1$

$\ln e^{3x} = \ln 10$

$$\log(9^{\checkmark}) = \log(5)$$

$$\frac{(7-3x)\log 9}{\log 9} = \frac{\log 5}{\log 9}$$

$$7 - 3x = \frac{\log 5}{\log 9}$$

$$-7 \qquad -7$$

$$\frac{-1}{3}(-3)x = -\frac{1}{3} \cdot \frac{\log 5}{\log 9} - 7\left(-\frac{1}{3}\right)$$

$$x = \boxed{\frac{-\log 5}{3\log 9} + \frac{7}{3}} \approx \boxed{2.09}$$

$\uparrow$ exact $\qquad$ approximation

$$\frac{3x}{3} = \frac{\ln(10)}{3}$$

$$x = \boxed{\frac{\ln 10}{3}}$$

steps

① Isolate exponential expression

② If possible, write both sides with same base, set exponents equal and solve.

→ ③ If ② fails, take "$\log_{10}$" or "$\ln$" of both sides, apply the power rule and solve.

(ex) Solve the logarithmic equation

a) $\log_3 x = \boxed{4}$

$$x = 3^4$$
$$\boxed{x = 81}$$

b) $\log_4 (x-3) = \log_4 (5)$

$$x - 3 = 5$$
$$\boxed{x = 8}$$

c) $\dfrac{\cancel{6} \cdot \log x}{\cancel{6}} = \dfrac{-10}{6}$

$$\log_{10} x = -\dfrac{5}{3}$$

$$x = 10^{-\frac{5}{3}}$$

$$x = \boxed{\dfrac{1}{10^{5/3}}}$$

d) $\overset{m}{\log (x-9)} \oplus \overset{N}{\log x} = 1$ $\underbrace{\log_a (mN)}_{} = \underbrace{\log_a m}_{} + \underbrace{\log_a N}_{}$

$$\log_{10} \left[(x-9) x \right] = 1$$

$$(x-9) x = 10^1$$

$$x^2 - 9x = 10$$

$$x^2 - 9x - 10 = 0$$
$$(x - 10)(x + 1) = 0$$
$$x - 10 = 0 \text{ or } x + 1 = 0$$
$$\boxed{x = 10} \text{ or } \cancel{x = -1}$$

extraneous

e) $\log_6 \overbrace{(x+3)}^{m} - \log_6 \overbrace{(x+2)}^{N} = \log_6 20$ $\left| \boxed{\log_a \dfrac{m}{N}} \right.$

$ \quad \underset{\leq 1.9}{\underbrace{}} \quad \underset{-1.2}{}$

$$= \log_a m - \log_a N$$

$$\log_6 \left(\frac{x+3}{x+2} \right) = \log_6 (20)$$

$$\frac{x+3}{x+2} = \frac{20}{1}$$

$$20(x+2) = x+3$$

$$20x + 40 = x + 3$$
$$ -x - 40 \quad -x - 40$$
$$\frac{19x}{19} = \frac{-37}{19}$$

$$\boxed{x = \frac{-37}{19}}$$

steps

① Get a single log on both sides, set inputs equal and solve.

or

② If ① fails, get a single log on one side, switch to exponential, and solve